

Points missed: _____

Student's Name: _____

Total score: _____/100 points

East Tennessee State University – Department of Computer and Information Sciences
CSCI 2710 (Tarnoff) – Discrete Structures
TEST 2 for Fall Semester, 2004

Read this before starting!

- This test is closed book and closed notes
- You may **NOT** use a calculator
- All answers must have a box drawn around them. This is to aid the grader (who might not be me!) Failure to do so might result in no credit for answer.
- If you perform work on the back of a page in this test, indicate that you have done so in case the need arises for partial credit to be determined.
- Statement regarding academic misconduct from Section 5.7 of the East Tennessee State University Faculty Handbook, June 1, 2001:

"Academic misconduct will be subject to disciplinary action. Any act of dishonesty in academic work constitutes academic misconduct. This includes plagiarizing, the changing or falsifying of any academic documents or materials, cheating, and the giving or receiving of unauthorized aid in tests, examinations, or other assigned school work. Penalties for academic misconduct will vary with the seriousness of the offense and may include, but are not limited to: a grade of 'F' on the work in question, a grade of 'F' of the course, reprimand, probation, suspension, and expulsion. For a second academic offense the penalty is permanent expulsion."

A short list of some tautologies:

- | | |
|---|--|
| 1. $(p \wedge q) \Rightarrow p$ | 2. $(p \wedge q) \Rightarrow q$ |
| 3. $p \Rightarrow (p \vee q)$ | 4. $q \Rightarrow (p \vee q)$ |
| 5. $\sim p \Rightarrow (p \Rightarrow q)$ | 6. $\sim(p \Rightarrow q) \Rightarrow p$ |
| 7. $((p \Rightarrow q) \wedge p) \Rightarrow q$ | 8. $((p \vee q) \wedge \sim p) \Rightarrow q$ |
| 9. $((p \Rightarrow q) \wedge \sim q) \Rightarrow \sim p$ | 10. $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$ |

Mathematical induction:

If $P(n_0)$ is true and assuming $P(k)$ is true implies $P(k+1)$ is true, then $P(n)$ is true for all $n \geq n_0$

Permutations and Combinations:

$${}_n P_r = \frac{n!}{(n-r)!} \qquad {}_n C_r = \frac{n!}{r!(n-r)!}$$

Properties of Relations:

- A relation is reflexive if $a R a$, for all $a \in A$.
- A relation is irreflexive if $a \not R a$, for all $a \in A$.
- A relation is symmetric if whenever $a R b$, then $b R a$.
- A relation is asymmetric if whenever $a R b$, then $b \not R a$.
- A relation is antisymmetric if whenever $a R b$ and $b R a$, then $a = b$.
- A relation is transitive if whenever $a R b$ and $b R c$, then $a R c$.
- A relation is called an equivalence relation if it is reflexive, symmetric, and transitive.

Each of the following six arguments uses one of the tautologies listed on the coversheet. (See table under the heading, "a short list of some tautologies.") For each of the four arguments, identify which tautology was used from this list *by entering a value 1 through 10 in the space provided*. (2 points each)

1.
$$\frac{\text{If it is thundering, then there is lightning}}{\text{There is thunder}} \\ \text{There is lightning}$$

Answer: 7

2.
$$\frac{\text{It is either raining or snowing}}{\text{It isn't raining}} \\ \text{It must be snowing}$$

Answer: 8

3.
$$\frac{\text{Either Ed is short or Ed is tall}}{\text{Ed is not short}} \\ \text{Ed is tall}$$

Answer: 8

4.
$$\frac{\text{If I drive to school, I will be late to class}}{\text{I am on time for class}} \\ \text{I didn't drive to school}$$

Answer: 9

5.
$$\frac{\text{This test is easy}}{\text{Either I studied well or this test is easy}}$$

Answer: 3 or 4

6.
$$\frac{\text{Matthew is my son}}{\text{Matthew is a child of mine}}$$

Answer: 1, 2, 3, or 4

For the next four arguments, *indicate which are valid and which are invalid*. (2 points each)

7.
$$\frac{\text{If I publish a novel, I will be famous}}{\text{If I am famous, I will be happy}} \\ \text{I am happy, therefore, I published a novel}$$

Valid Invalid

8.
$$\frac{\text{If I drive to school, I will be late to class}}{\text{I was late to class}} \\ \text{I drove to school}$$

Valid Invalid

9.
$$\frac{\text{If I try hard, then I will succeed}}{\text{If I succeed, then I will be happy}} \\ \text{I am not happy, therefore, I didn't try hard}$$

Valid Invalid

10.
$$\frac{\text{Pete is the name of my pet}}{\text{The only dogs I own are black labs}} \\ \text{Pete is a black lab}$$

Valid Invalid

The following seven problems present seven situations where r items are selected from a set of n items. **Select the formula, n^r , ${}_nP_r$, ${}_nC_r$, or ${}_{(n+r-1)}C_r$, that will compute the number of different, valid sequences and identify the values of r and n .** (4 points each)

To answer these problems, you simply need to remember which formula pertains to which situation: ordered/unordered and duplicates allowed/duplicates not allowed. Each situation is either ordered with duplicates allowed, n^r , ordered with no duplicates allowed, ${}_nP_r$, unordered with no duplicates allowed, ${}_nC_r$, or unordered with duplicates allowed, ${}_{(n+r-1)}C_r$. From there, it's just a matter of setting n to the number of items in the set being selected from and r to the number of items being selected.

11. Compute the number of 4-digit ATM PINs where duplicate digits are allowed.

a.) n^r b.) ${}_nP_r$ c.) ${}_nC_r$ d.) ${}_{(n+r-1)}C_r$ $n = \underline{10}$ $r = \underline{4}$

12. Compute the number of different 5 card hands can be drawn from a deck of 52 cards.
 a.) n^r b.) ${}_nP_r$ **c.) ${}_nC_r$** d.) ${}_{(n+r-1)}C_r$ $n = \underline{52}$ $r = \underline{5}$
13. How many committees of 5 people can be created from a group of 8 people?
 a.) n^r b.) ${}_nP_r$ **c.) ${}_nC_r$** d.) ${}_{(n+r-1)}C_r$ $n = \underline{8}$ $r = \underline{5}$
14. How many ways can the letters in the word MICHAEL be arranged?
 a.) n^r **b.) ${}_nP_r$** c.) ${}_nC_r$ d.) ${}_{(n+r-1)}C_r$ $n = \underline{7}$ $r = \underline{7}$
15. Assume you need to buy 10 bottles of soda from a selection of {Coke, Pepsi, Dr. Pepper, and Sprite}. How many ways could you do this?
 a.) n^r b.) ${}_nP_r$ c.) ${}_nC_r$ **d.) ${}_{(n+r-1)}C_r$** $n = \underline{4}$ $r = \underline{10}$
16. How many three-digit numbers are there in base-5? Assume leading zeros are included as digits.
a.) n^r b.) ${}_nP_r$ c.) ${}_nC_r$ d.) ${}_{(n+r-1)}C_r$ $n = \underline{5}$ $r = \underline{3}$
17. How many different ways can 2 six-sided dice come up? There is no order, e.g., 3 & 4 are the same as 4 & 3.
 a.) n^r b.) ${}_nP_r$ c.) ${}_nC_r$ **d.) ${}_{(n+r-1)}C_r$** $n = \underline{6}$ $r = \underline{2}$
18. **True** or false: In selecting r items from a set of n items where order doesn't matter and duplicates are allowed, r may be greater than n . (2 points)
19. Which of the following expressions describes how to calculate the number of available license plate combinations of the format "ABC 123"? (2 points)
 a.) ${}_{26}C_3 \cdot {}_{10}C_3$ **b.) ${}_{(26+10-1)}C_{10}$** c.) ${}_{26}P_3 \cdot {}_{10}P_3$ d.) ${}_{(26+10-1)}P_6$
 e.) ${}_{(36+6-1)}C_6$ **f.) $26^3 \cdot 10^3$** g.) ${}_{26}C_3 \cdot 10^3$ h.) None of the above
20. Assume we have a lottery where you first pick 5 from a group of 60 then pick one powerball option from a group of 45? What is the ratio of picking the wrong powerball to picking the right powerball? (2 points)
 a.) 45:1 **b.) 44:1** c.) ${}_{60}C_5 \cdot 44 : {}_{60}C_5 \cdot 45$ d.) ${}_{45}C_1 : 1$ f.) None of the above
21. Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. List all of the elements in $A \times B$. (3 points)
 $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$
22. If $|A| = 5$ and $|B| = 10$, then the cardinality of $A \times B$ is: (2 points)
a.) $5 \cdot 10$ b.) 10^5 c.) 5^{10} d.) ${}_5C_{10}$ e.) ${}_5P_{10}$ f.) None of the above

The next 5 problems represent relations across the Cartesian product $A \times A$ where $A = \{a, b, c, d\}$. The relations are represented either as subsets of $A \times A$, matrices, or digraphs. For each problem, determine whether the relation is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, and/or transitive. **Check all that apply.** (4 points each)

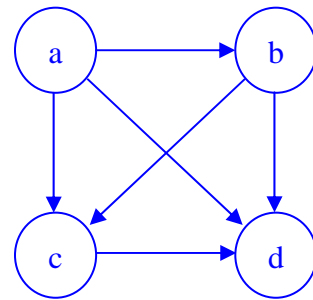
For each of the following answers, either the matrix or the digraph is used to determine the relation's characteristics.

- Reflexive says that there must be ones all along the main diagonal of the matrix and every vertex in the digraph has an edge that loops back to itself.
- Irreflexive says that there must be zeros all along the main diagonal of the matrix and no vertex in the digraph has an edge that loops back to itself.
- Symmetric says that the matrix is symmetric across the main diagonal and for every edge in the digraph, there is an equal edge going back the other way, i.e., every edge is bidirectional (cycles of length one are allowed).
- Asymmetric says that for every '1' in the matrix, there must be a '0' opposite the main diagonal from it and there can be no 1's on the diagonal while every edge in the digraph must be only one-way and there can be no loop backs, i.e., cycles of length one.
- Antisymmetric says that for every '1' in the matrix, there must be a '0' opposite the main diagonal from it, but 1's are allowed on the diagonal. Every edge in the digraph must be only one-way, but loop backs, i.e., cycles of length one, are allowed.
- Transitive says that for every pair of 1's in a matrix a_{ij} and a_{jk} , there must be a '1' at a_{ik} . The digraph says that for every path of length two, there must also be a path of length one.

23. $R = \{(a,b), (a,c), (a,d), (b,c), (b,d), (c,d)\}$

Below, I have created both the matrix defining R and the digraph.

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

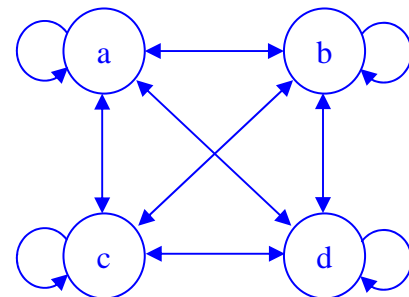


- reflexive irreflexive symmetric asymmetric antisymmetric transitive

24. $R = A \times A$

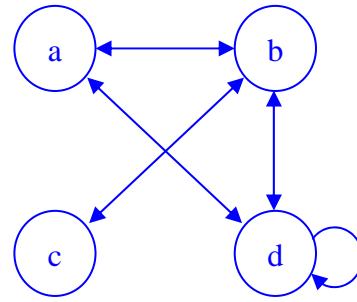
Below, I have created both the matrix defining R and the digraph.

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



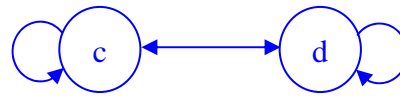
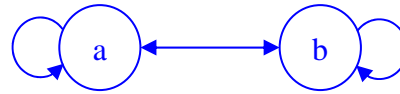
- reflexive irreflexive symmetric asymmetric antisymmetric transitive

25.
$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

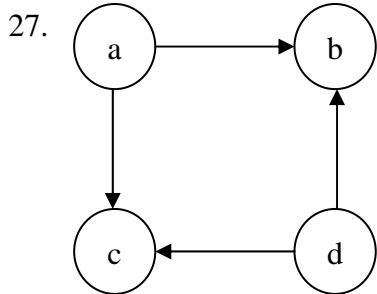


- reflexive irreflexive symmetric asymmetric antisymmetric transitive

26.
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$



- reflexive irreflexive symmetric asymmetric antisymmetric transitive



$$M_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

- reflexive irreflexive symmetric asymmetric antisymmetric transitive

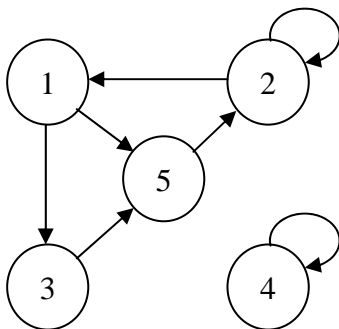
The next three problems represent relations across the Cartesian product $A \times A$ where $A = \{1, 2, 3, 4, 5\}$.

28. Write the set of ordered pairs represented by the relation matrix (4 points)

$$R = \{(1, 1), (1, 3), (2, 1), (2, 5), (3, 2), (4, 5), (5, 2), (5, 5)\}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

29. Convert the following digraph to a matrix. (3 points)

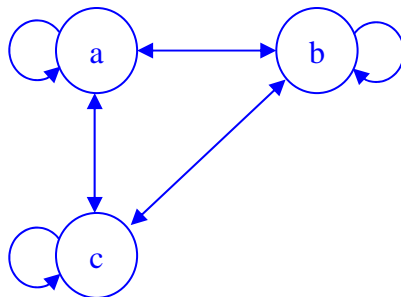


$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

30. Fill out the table below listing the in-degree and out degree of each element for the digraph of the previous problem. (4 points)

	1	2	3	4	5
In-Degree	1	2	1	1	2
Out-Degree	2	2	1	1	1

31. Create the digraph of the relation $R = A \times A$ for the set $A = \{a, b, c\}$. (3 points)



32. **Select only one** of the following statements to prove true using mathematical induction. (7 points)

a.) $2 + 4 + 6 + \dots + 2n = n(n + 1)$

b.) $1 + 2^1 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 1$

c.) $1 + a^1 + a^2 + a^3 + \dots + a^{n-1} = \frac{a^n - 1}{a - 1}$

a.) First, test to see if the base case is true, i.e., the $n=1$ case:

$$1 \cdot (1+1) = 1 \cdot 2 = 2 \rightarrow \text{this case is TRUE!}$$

Now, assume that the k case is true. The k case looks like this:

$$2 + 4 + 6 + \dots + 2k = k(k + 1)$$

What we want to do is make this look like the $k+1$ case, i.e., what we are trying to prove is $2 + 4 + 6 + \dots + 2k + 2(k + 1) = (k + 1)(k + 2)$. To get the k case to look like this, we need to begin by adding $2(k+1)$ to both sides of the k expression above.

$$2 + 4 + 6 + \dots + 2k + 2(k + 1) = k(k + 1) + 2(k + 1)$$

On the right side of the equation, pulling a $(k + 1)$ from both of the product terms gives us:

$$2 + 4 + 6 + \dots + 2k + 2(k + 1) = (k + 1)(k + 2)$$

And this proves that the expression $2 + 4 + 6 + \dots + 2n = n(n + 1)$ is true for all $n \geq 1$.

b.) First, test to see if the base case is true, i.e., the $n=1$ case:

$$1 + 2^1 = 3 = 2^{1+1} - 1 = 2^2 - 1 = 4 - 1 = 3 \rightarrow \text{this case is TRUE!}$$

Now, assume that the k case is true. The k case looks like this:

$$1 + 2^1 + 2^2 + 2^3 + \dots + 2^k = 2^{k+1} - 1$$

What we want to do is make this look like the $k+1$ case, i.e., what we are trying to prove is

$1 + 2^1 + 2^2 + 2^3 + \dots + 2^{k+1} = 2^{k+2} - 1$. To get the k case to look like this, we need to begin by adding 2^{k+1} to both sides of the k expression above.

$$1 + 2^1 + 2^2 + 2^3 + \dots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$

$$1 + 2^1 + 2^2 + 2^3 + \dots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$$

$$1 + 2^1 + 2^2 + 2^3 + \dots + 2^k + 2^{k+1} = 2 \cdot 2^{k+1} - 1$$

$$1 + 2^1 + 2^2 + 2^3 + \dots + 2^k + 2^{k+1} = 2^{k+1+1} - 1$$

$$1 + 2^1 + 2^2 + 2^3 + \dots + 2^k + 2^{k+1} = 2^{k+2} - 1$$

And this proves that the expression $1 + 2^1 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 1$ is true for all $n \geq 1$.

c.) First, test to see if the base case is true, i.e., the $n=1$ case:

$$(a^1 - 1)/(a - 1) = (a - 1)/(a - 1) = 1 \rightarrow \text{this case is TRUE!}$$

Now, assume that the k case is true. The k case looks like this:

$$1 + a^1 + a^2 + a^3 + \dots + a^{k-1} = (a^k - 1)/(a - 1)$$

What we want to do is make this look like the $k+1$ case, i.e., what we are trying to prove is

$1 + a^1 + a^2 + a^3 + \dots + a^{k-1} + a^k = (a^{k+1} - 1)/(a - 1)$. To get the k case to look like this, we need to begin by adding a^k to both sides of the k expression above.

$$1 + a^1 + a^2 + a^3 + \dots + a^{k-1} + a^k = (a^k - 1)/(a - 1) + a^k$$

Now, multiply the a^k term on the right side of the equation by $(a - 1)/(a - 1)$ so that we can combine it with the first term.

$$1 + a^1 + a^2 + a^3 + \dots + a^{k-1} + a^k = (a^k - 1)/(a - 1) + a^k \cdot (a - 1)/(a - 1) = [(a^k - 1) + a^k \cdot (a - 1)]/(a - 1)$$

Next, multiply the term $a^k \cdot (a - 1)$ through

$$1 + a^1 + a^2 + a^3 + \dots + a^{k-1} + a^k = [(a^k - 1) + a^{k+1} - a^k]/(a - 1)$$

$$1 + a^1 + a^2 + a^3 + \dots + a^{k-1} + a^k = [a^k - a^k - 1 + a^{k+1}]/(a - 1) = (a^{k+1} - 1)/(a - 1)$$

And this proves that the expression $1 + a^1 + a^2 + a^3 + \dots + a^{n-1} = (a^n - 1)/(a - 1)$ is true for all $n \geq 1$.