Points missed: _____ Student's Name: ____

Total score: _____/100 points

East Tennessee State University – Department of Computer and Information Sciences
CSCI 1900 (Tarnoff) – Discrete Structures
TEST 2 for Summer Semester, 2005

Read this before starting!

- This test is closed book and closed notes
- You may ***NOT*** use a calculator
- All answers must have a box drawn around them. This is to aid the grader (who might not be me!) Failure to do so might result in no credit for answer.
- If you perform work on the back of a page in this test, indicate that you have done so in case the need arises for partial credit to be determined.
- Statement regarding academic misconduct from Section 5.7 of the East Tennessee State University Faculty Handbook, June 1, 2001:

  "Academic misconduct will be subject to disciplinary action. Any act of dishonesty in academic work constitutes academic misconduct. This includes plagiarism, the changing of falsifying of any academic documents or materials, cheating, and the giving or receiving of unauthorized aid in tests, examinations, or other assigned school work. Penalties for academic misconduct will vary with the seriousness of the offense and may include, but are not limited to: a grade of 'F' on the work in question, a grade of 'F' of the course, reprimand, probation, suspension, and expulsion. For a second academic offense the penalty is permanent expulsion."

A short list of some tautologies:

1. $(p \wedge q) \Rightarrow p$
2. $(p \wedge q) \Rightarrow q$
3. $p \Rightarrow (p \vee q)$
4. $q \Rightarrow (p \vee q)$
5. $\sim p \Rightarrow (p \Rightarrow q)$
6. $\sim(p \Rightarrow q) \Rightarrow p$
7. $((p \Rightarrow q) \wedge p) \Rightarrow q$
8. $((p \vee q) \wedge \sim p) \Rightarrow q$
9. $((p \Rightarrow q) \wedge \sim q) \Rightarrow \sim p$
10. $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$

Mathematical induction:

If $P(n_0)$ is true and assuming $P(k)$ is true implies $P(k+1)$ is true, then $P(n)$ is true for all $n \geq n_0$

Permutations and Combinations:

$${}_nP_r = \frac{n!}{(n - r)!} \qquad {}_nC_r = \frac{n!}{r!(n - r)!}$$

Properties of operations for propositions

Commutative Properties

1. $p \vee q \equiv q \vee p$
2. $p \wedge q \equiv q \wedge p$

Associative Properties

3. $p \vee (q \vee r) \equiv (p \vee q) \vee r$
4. $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

Distributive Properties

5. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
6. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Idempotent Properties

7. $p \vee p \equiv p$
8. $p \wedge p \equiv p$

Properties of Negation

9. $\sim(\sim p) \equiv p$
10. $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$
11. $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

Short answers – 2 points each unless otherwise noted

For problems 1 through 4, indicate whether the phrase is a statement or not.

1. "Study hard!" ☐ Statement ☑ Not a statement
2. "How hard did you study for this test?" ☐ Statement ☑ Not a statement
3. "42 is the answer." ☑ Statement ☐ Not a statement
4. "2 is an odd number." ☑ Statement ☐ Not a statement
5. Give the negation of the statement "2 + 7 = 9."

$2 + 7 \neq 9$

6. Give the negation of the statement "It will rain tomorrow or it will snow tomorrow." (3 points)

It will not rain tomorrow and it will not snow tomorrow.

For problems 7 and 8, find the truth value of each proposition if ***p* and *q* are true** and ***r* is false**.

7. $p \wedge \sim q \wedge \sim r$ ☐ True ☑ False

Substituting T for *p* and *q* and F for *r* gives us:

$T \wedge \sim T \wedge \sim F = T \wedge F \wedge T = F$

8. $p \wedge (r \vee q)$ ☑ True ☐ False

Substituting T for *p* and *q* and F for *r* gives us:

$T \wedge (F \vee T) = T \wedge T = T$

For problems 9 and 10, convert the sentence given to an expression in terms of p, q, r, and logical connectives ($\sim$, $\vee$, $\wedge$, $\Leftrightarrow$, and $\Rightarrow$) if p: I'm rich; q: I work hard; and r: I'm lucky.

9. I'm not lucky, but I work hard. Answer: $\sim r \wedge q$

Number 9 was tricky in that you had to figure out what substituted for "but". By substituting "and", "or", "if and only if", and "implies" for "but", we see that the only one that keeps the meaning of the sentence is "and". I'm not lucky, and I work hard.

10. If I work hard and I'm lucky, then I will be rich. Answer: $(q \wedge r) \Rightarrow p$

Each of the following six arguments uses one of the tautologies listed on the coversheet. (See table under the heading, "a short list of some tautologies.") For each of the four arguments, identify which tautology was used from this list ***by entering a value 1 through 10 in the space provided***.

11.

If Tarnoff is lecturing, I'm fascinated	$p \Rightarrow q$
If I'm fascinated, then I'm awake	$q \Rightarrow r$
If Tarnoff is lecturing, then I'm awake	$p \Rightarrow r$

Answer: **10**

12.

If I studied, I will pass this test	$p \Rightarrow q$
I studied	p
I will pass this test	q

Answer: **7**

13. If I like country music, then I like Alan Jackson
I don't like Alan Jackson $p \Rightarrow q$
I must not like country music $\sim q$
$\sim p$

Answer: **9**

14. I have a child that is a girl or a boy $p \vee q$
I do not have a daughter $\sim q$
I must have a son p

Answer: **8**

15. If I finish studying, I will go watch a movie
I finished studying $p \Rightarrow q$
I will go watch a movie p
q

Answer: **7**

16. This test is easy p
Either this test is easy or I studied $p \vee q$

Answer: **4 or 3**

For the next four arguments, ***indicate which are valid and which are invalid***.

17. If I walk to school, I am tired
I am tired
I must have walked to school

☐ Valid ☑ Invalid

18. If it rains, I will not walk to school
I walked to school
It isn't raining

☑ Valid ☐ Invalid

19. If I win the lottery, I will invest wisely
If I invest wisely, I will be rich
I won the lottery, therefore, I am rich

☑ Valid ☐ Invalid

20. If I own a dog, it will be a male dog
Spot is my pet
Spot is a male dog

☐ Valid ☑ Invalid

Each of these arguments can be proven or disproven by coming up with the expression they represent and then showing whether the expression is a tautology or not. You can also show that an argument is invalid by coming up with an example showing when it might not be true.

- In 17, it is possible that I'm tired because I didn't get any sleep. The expression the argument represents is $((p \Rightarrow q) \wedge q) \Rightarrow p$. This is not a tautology.

p	q	$(p \Rightarrow q)$	$((p \Rightarrow q) \wedge q)$	$((p \Rightarrow q) \wedge q) \Rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

- In 18, this is the tautology $((p \Rightarrow q) \wedge \sim q) \Rightarrow \sim p$.
- Since I can find a reason whyif Casey is the name of my pet, and the only pets I own are dogs, then Casey must be a dog.
- In 19, this is the tautology $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$.
- In 20, it is possible that Spot is a goldfish or a cat, still pets. The expression the argument represents is $((p \Rightarrow q) \wedge r) \Rightarrow s$. Don't even bother with the truth table, the statements are so mixed up that there's nothing to prove.

The following seven problems present seven situations where r items are selected from a set of n items. **Select the formula, n^r, ${}_nP_r$, ${}_nC_r$, or ${}_{(n+r-1)}C_r$, that will compute the number of different, valid sequences and identify the values of r and n.** (4 points each)

To answer these problems, you simply need to remember which formula pertains to which situation: ordered/unordered and duplicates allowed/duplicates not allowed. Each situation is either ordered with duplicates allowed, n^r, ordered with no duplicates allowed, ${}_nP_r$, unordered with no duplicates allowed, ${}_nC_r$, or unordered with duplicates allowed, ${}_{(n+r-1)}C_r$. From there, it's just a matter of setting n to the number of items in the set being selected from and r to the number of items being selected.

21. Compute the number of possible four letter words using the English alphabet including the nonsensical ones like "zzyq".

    (a.) n^r (circled)   b.) ${}_nP_r$ c.) ${}_nC_r$,   d.) ${}_{(n+r-1)}C_r$ n = __26__ r = __4__

22. How many ways can you create a 5 person committee from a group of 27 employees?

    a.) n^r b.) ${}_nP_r$ (c.) ${}_nC_r$, (circled)   d.) ${}_{(n+r-1)}C_r$ n = __27__ r = __5__

23. How many subsets are there of the set B = {1, 2, 3, 4, 5, 6}?

    This is that tricky one we did in class. Basically, each element is either a member of the subset or not a member of the subset. This can be represented with a five digit binary number. For example, the binary number 111111 would represent the subset that contained all elements of B, i.e., {1, 2, 3, 4, 5, 6}. The binary number 101010 would represent the subset {1, 3, 5}. Therefore, since there are 2^6 possible 6-digit binary numbers, the answer is 2^6.

    (a.) n^r (circled)   b.) ${}_nP_r$ c.) ${}_nC_r$,   d.) ${}_{(n+r-1)}C_r$ n = __2__ r = __6__

24. How many 7-digit numbers are there in base-3? Assume leading zeros are included as digits.

    (a.) n^r (circled)   b.) ${}_nP_r$ c.) ${}_nC_r$,   d.) ${}_{(n+r-1)}C_r$ n = __3__ r = __7__

25. How many ways can 35 drivers finish first, second, and third in a race in a specific order? The rest of the finishing order is unimportant.

    a.) n^r (b.) ${}_nP_r$ (circled)   c.) ${}_nC_r$,   d.) ${}_{(n+r-1)}C_r$ n = __35__ r = __3__

26. How many ways can 35 drivers finish first, second, and third in a race in any order? The rest of the finishing order is unimportant.

    a.) n^r b.) ${}_nP_r$ (c.) ${}_nC_r$, (circled)   d.) ${}_{(n+r-1)}C_r$ n = __35__ r = __3__

27. How many ways can 4 types of chocolate be used to fill a box with slots for 8 pieces of chocolate?

    a.) n^r b.) ${}_nP_r$ c.) ${}_nC_r$,   (d.) ${}_{(n+r-1)}C_r$ (circled) n = __4__ r = __8__

28. True or (false) (circled): r must always be less than or equal to n when determining the number of ways r items can be selected from a set of n items when order matters and duplicates are allowed.

    False: r must be less than or equal to n only when duplicates are not allowed.

29. True or false: $_nP_1 = {}_nC_1 = n^1$.

True: There are two ways to do this. First, you can reason it out. Each of these formulas says that we are picking 1 element from a set of *n* elements. If you are only picking out one element from a set of *n* elements, how can order or duplicates matter? It's the same for all of them.

The second way to do it is to simply work out the formulas. (Note that they are given to you on the front page.)

$$_nP_1 = \frac{n!}{(n-1)!} = \frac{n\times(n-1)\times(n-2)\times\ldots\times2\times1}{(n-1)\times(n-2)\times\ldots\times2\times1} = n$$

$$_nC_1 = \frac{n!}{1!(n-1)!} = \frac{n\times(n-1)\times(n-2)\times\ldots\times2\times1}{1\times(n-1)\times(n-2)\times\ldots\times2\times1} = n$$

$$n^1 = n$$

30. Assume that a computer can come configured with one of 3 different processors and one of 5 different memory configurations. Which of the following expressions describes how to calculate the number of ways that this computer can be configured?

a.) $_{(3+5-1)}C_2$ b.) $3 \cdot 5$ c.) $_8P_2$ d.) $_8C_2$ h.) None of the above

31. Remember that the Powerball lottery game consists of selecting 5 numbers from 53 and 1 Powerball number from 42. Select which of the following expressions describes how to calculate the number of ways that exactly 4 of the 5 numbers have been selected correctly and the Powerball number was selected incorrectly.

Begin by looking at the pick 5 portion of the problem in the following way: we must first pick one of the 5 right values to be wrong ($_5C_1$), then pick one of the remaining 48 values to set the wrong value to ($_{48}C_1$). This gives us $_{48}C_1 \cdot {}_5C_1$. Then we must pick one of the 41 remaining values of the Powerball for our wrong Powerball selection (41). Multiplying these together gives us $_{48}C_1 \cdot {}_5C_1 \cdot 41$.

a.) ~~$_{48}C_1 \cdot 41$~~ b.) $_{48}P_1 \cdot 41$ c.) $_{53}C_1 \cdot 42$ d.) $_{53}P_1 \cdot 42$
e.) $_{48}C_1 \cdot {}_5C_1 \cdot 41$ f.) $_{48}P_1 \cdot {}_5P_1$ g.) $_{53}C_1 \cdot {}_5C_1 \cdot 42$ h.) None of the above

32. What is the probability that you will get a heads from a single flip of a quarter? (Assume all outcomes are equally likely.)

Total possible outcomes = 2, one for each side.
Successful outcomes = 1
Probability = successful outcomes/total possible outcomes = ½

a.) 0 b.) 1/4 c.) 1/3 d.) 1/2 e.) 1 f.) None of the above

33. What is the probability that you will get ***at least one*** heads from three flips of a quarter? (Assume all outcomes are equally likely.)

Total possible outcomes = 8, (H-H-H, H-H-T, H-T-H, H-T-T, T-H-H, T-H-T, T-T-H, or T-T-T)
Successful outcomes = 7, (H-H-H, H-H-T, H-T-H, H-T-T, T-H-H, T-H-T, or T-T-H)
Probability = successful outcomes/total possible outcomes = 7/8

a.) 1/3 b.) 2/3 c.) 2/4 d.) 3/4 e.) 5/8 f.) 7/8 g.) 1

34. What is the probability that you will get a '5' from a single roll of a single six-sided die? (Assume all outcomes are equally likely.)

Total possible outcomes = 6, ('1', '2', '3', '4', '5', or '6')
Successful outcomes = 1
Probability = successful outcomes/total possible outcomes = 1/6

a.) 0 b.) 1/5 c.) 5/6 d.) 1/3 e.) 1/6 f.) None of the above

35. What is the probability that you will get an even number from a single roll of a single six-sided die? (Assume all outcomes are equally likely.)

Total possible outcomes = 6, ('1', '2', '3', '4', '5', or '6')
Successful outcomes = 3, ('2', '4', or '6')
Probability = successful outcomes/total possible outcomes = 3/6

a.) 1/6 b.) 3/5 c.) 4/6 d.) $6/6^2$ e.) 3/6 f.) 5/6 g.) 1

Medium answers – 4 points each unless otherwise noted

36. Use truth tables to show that $\sim(p \Rightarrow q) \Rightarrow p$ is a tautology. Show ***all*** intermediate steps. Be sure to label columns.

p	q	$(p \Rightarrow q)$	$\sim(p \Rightarrow q)$	p (duplicate)	$\sim(p \Rightarrow q) \Rightarrow p$
T	T	T	F	T	T
T	F	F	T	T	T
F	T	T	F	F	T
F	F	T	F	F	T

37. Use truth tables to show that $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$ is a tautology. Show ***all*** intermediate steps. Be sure to label columns.

Remember to swap a $\Leftrightarrow$ for a $\equiv$ when proving the tautology. Once you've shown that it's a tautology with the $\Leftrightarrow$ symbol, you've proven that it is an equivalence and can go back to the $\equiv$ symbol.

p	q	$(p \wedge q)$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$	$\sim(p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

Mathematical induction problem – 7 points

38. ***Select only one*** of the following statements to prove true using mathematical induction.

a.) $1 + 2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$

b.) $2 + 4 + 6 + \ldots + 2n = n(n + 1)$

c.) $5 + 10 + 15 + \ldots + 5n = \dfrac{5n(n+1)}{2}$

a.) First, test to see if the base case is true, i.e., the n=1 case:

$1 + 2^1 = 3 = 2^{1+1} - 1 = 2^2 - 1 = 4 - 1 = 3 \rightarrow$ this case is TRUE!

Now, assume that the k case is true. The k case looks like this:

$1 + 2^1 + 2^2 + 2^3 + \ldots + 2^k = 2^{k+1} - 1$

What we want to do is make this look like the k+1 case, i.e., what we are trying to prove is $1 + 2^1 + 2^2 + 2^3 + \ldots + 2^{k+1} = 2^{k+2} - 1$. To get the k case to look like this, we need to begin by adding 2^{k+1} to both sides of the k expression above.

$1 + 2^1 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$

$1 + 2^1 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$

$1 + 2^1 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} = 2 \cdot 2^{k+1} - 1$

$1 + 2^1 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} = 2^{k+1+1} - 1$

$1 + 2^1 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

And this proves that the expression $1 + 2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$ is true for all $n \geq 1$.

b.) $2 + 4 + 6 + \ldots + 2n = n(n + 1)$

Basis case: n = 1
$2 = 1 \cdot (1 + 1) = 2$ ← It works for n = 1!

Assume the k case is true:
$2 + 4 + 6 + \ldots + 2k = k(k + 1)$

From it, derive the k+1 case which is $2 + 4 + 6 + \ldots + 2k + 2(k+1) = (k + 1)\cdot(k + 1 + 1)$

$2 + 4 + 6 + \ldots + 2k + 2(k + 1) = k(k + 1) + 2(k + 1)$ — Add 2(k + 1) to both sides

$= (k + 1)\cdot(k + 2)$ — Pull out the (k + 1) from both terms

$= (k + 1)\cdot(k + 1 + 1)$ — Set 2 equal to 1 + 1.

Since the last line equals the k+1 case, we've proven the formula for all values $n \geq 1$.

c.) $5 + 10 + 15 + \ldots + 5n = \dfrac{5n(n+1)}{2}$

Basis case: n = 1

$5 = \dfrac{5 \cdot 1 \cdot (1+1)}{2} = 5$ ← It works for n = 1!

Assume the k case is true:

$$5 + 10 + 15 + \ldots + 5k = \frac{5k(k+1)}{2}$$

From the k case, prove the k+1 case which is shown below:

$$5 + 10 + 15 + \ldots + 5k + 5(k+1) = \frac{5(k+1)(k+1+1)}{2}$$

$$5 + 10 + 15 + \ldots + 5k + 5(k+1) = \frac{5k(k+1)}{2} + 5(k+1)$$ ← Add 5(k + 1) to both sides.

$$5 + 10 + 15 + \ldots + 5k + 5(k+1) = \frac{5k(k+1)}{2} + \frac{10(k+1)}{2}$$

$$5 + 10 + 15 + \ldots + 5k + 5(k+1) = \frac{5k(k+1) + 10(k+1)}{2}$$

Pull out 5 and (k + 1)

$$5 + 10 + 15 + \ldots + 5k + 5(k+1) = \frac{5(k+1)[k+2]}{2}$$

$$5 + 10 + 15 + \ldots + 5k + 5(k+1) = \frac{5(k+1)(k+1+1)}{2}$$

Since the last line equals the k+1 case, we've proven the formula for all values $n \geq 1$.