CS 373: Combinatorial Algorithms, Fall 2000

Homework 0, due August 31, 2000 at the beginning of class

Name:	
Net ID:	Alias:

Neatly print your name (first name first, with no comma), your network ID, and a short alias into the boxes above. **Do not** *sign* **your name. Do** *not* **write your Social Security number.** Staple this sheet of paper to the top of your homework.

Grades will be listed on the course web site by alias give us, so your alias should not resemble your name or your Net ID. If you don't give yourself an alias, we'll give you one that you won't like.

Before you do anything else, read the Homework Instructions and FAQ on the CS 373 course web page (http://www-courses.cs.uiuc.edu/~cs373/hw/faq.html), and then check the box below. This web page gives instructions on how to write and submit homeworks—staple your solutions together in order, write your name and netID on every page, don't turn in source code, analyze everything, use good English and good logic, and so forth.

☐ I have read the CS 373 Homework Instructions and FAQ.

This homework tests your familiarity with the prerequisite material from CS 173, CS 225, and CS 273—many of these problems have appeared on homeworks or exams in those classes—primarily to help you identify gaps in your knowledge. **You are responsible for filling those gaps on your own.** Parberry and Chapters 1–6 of CLR should be sufficient review, but you may want to consult other texts as well.

Required Problems

1. Sort the following 25 functions from asymptotically smallest to asymptotically largest, indicating ties if there are any:

1	n	n^2	$\lg n$	$\lg(n \lg n)$
$\lg^* n$	$\lg^* 2^n$	$2^{\lg^* n}$	$\lg \lg^* n$	$\lg^* \lg n$
$n^{\lg n}$	$(\lg n)^n$	$(\lg n)^{\lg n}$	$n^{1/\lg n}$	$n^{\lg \lg n}$
$\log_{1000} n$	$\lg^{1000} n$	$\lg^{(1000)} n$	$\left(1+\frac{1}{n}\right)^n$	$n^{1/1000}$

   To simplify notation, write $f(n) \ll g(n)$ to mean $f(n) = o(g(n))$ and $f(n) \equiv g(n)$ to mean $f(n) = \Theta(g(n))$. For example, the functions n^2, n, $\binom{n}{2}$, n^3 could be sorted either as $n \ll n^2 \equiv \binom{n}{2} \ll n^3$ or as $n \ll \binom{n}{2} \equiv n^2 \ll n^3$.

2. (a) Prove that any positive integer can be written as the sum of distinct powers of 2. For example: $42 = 2^5 + 2^3 + 2^1$, $25 = 2^4 + 2^3 + 2^0$, $17 = 2^4 + 2^0$. [Hint: "Write the number in binary" is *not* a proof; it just restates the problem.]

   (b) Prove that any positive integer can be written as the sum of distinct *nonconsecutive* Fibonacci numbers—if F_n appears in the sum, then neither F_{n+1} nor F_{n-1} will. For example: $42 = F_9 + F_6$, $25 = F_8 + F_4 + F_2$, $17 = F_7 + F_4 + F_2$.

   (c) Prove that *any* integer (positive, negative, or zero) can be written in the form $\sum_i \pm 3^i$, where the exponents i are distinct non-negative integers. For example: $42 = 3^4 - 3^3 - 3^2 - 3^1$, $25 = 3^3 - 3^1 + 3^0$, $17 = 3^3 - 3^2 - 3^0$.

3. Solve the following recurrences. State tight asymptotic bounds for each function in the form $\Theta(f(n))$ for some recognizable function $f(n)$. You do not need to turn in proofs (in fact, please *don't* turn in proofs), but you should do them anyway just for practice. If no base cases are given, assume something reasonable but nontrivial. Extra credit will be given for more exact solutions.

   (a) $A(n) = 3A(n/2) + n$

   (b) $B(n) = \max_{n/3<k<2n/3} \big(B(k) + B(n-k) + n\big)$

   (c) $C(n) = 4C(\lfloor n/2 \rfloor + 5) + n^2$

   *(d) $D(n) = 2D(n/2) + n/\lg n$

   *(e) $E(n) = \dfrac{E(n-1)}{E(n-2)}$, where $E(1) = 1$ and $E(2) = 2$.

4. Penn and Teller have a special deck of fifty-two cards, with no face cards and nothing but clubs—the ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ..., 52 of clubs. (They're big cards.) Penn shuffles the deck until each each of the 52! possible orderings of the cards is equally likely. He then takes cards one at a time from the top of the deck and gives them to Teller, stopping as soon as he gives Teller the three of clubs.

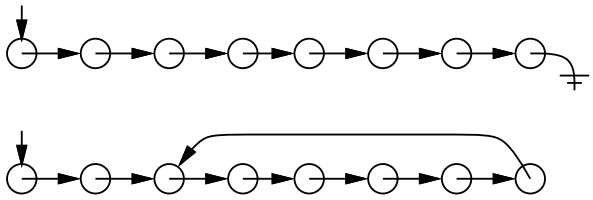
   (a) On average, how many cards does Penn give Teller?

   (b) On average, what is the smallest-numbered card that Penn gives Teller?

   *(c) On average, what is the largest-numbered card that Penn gives Teller?

   [Hint: Solve for an n-card deck, and then set $n = 52$.] Prove that your answers are correct. If you have to appeal to "intuition" or "common sense", your answers are probably wrong!

5. Suppose you have a pointer to the head of singly linked list. Normally, each node in the list only has a pointer to the next element, and the last node's pointer is NULL. Unfortunately, your list might have been corrupted by a bug in somebody else's code[1], so that the last node has a pointer back to some other node in the list instead.

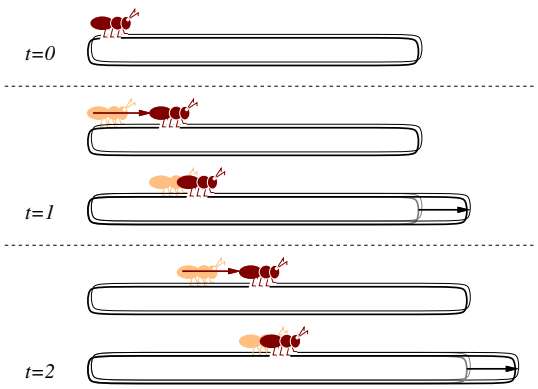


Top: A standard singly-linked list. Bottom: A corrupted singly linked list.

Describe an algorithm[2] that determines whether the linked list is corrupted or not. Your algorithm must not modify the list. For full credit, your algorithm should run in $O(n)$ time, where n is the number of nodes in the list, and use $O(1)$ extra space (not counting the list itself).

6. *[This problem is required only for graduate students taking CS 373 for a full unit; anyone else can submit a solution for extra credit.]*

An ant is walking along a rubber band, starting at the left end. Once every second, the ant walks one inch to the right, and then you make the rubber band one inch longer by pulling on the right end. The rubber band stretches uniformly, so stretching the rubber band also pulls the ant to the right. The initial length of the rubber band is n inches, so after t seconds, the rubber band is $n + t$ inches long.



Every second, the ant walks an inch, and then the rubber band is stretched an inch longer.

(a) How far has the ant moved after t seconds, as a function of n and t? Set up a recurrence and (for full credit) give an *exact* closed-form solution. [Hint: What *fraction* of the rubber band's length has the ant walked?]

*(b) How long does it take the ant to get to the right end of the rubber band? For full credit, give an answer of the form $f(n) + \Theta(1)$ for some explicit function $f(n)$.

[1] After all, *your* code is always completely 100% bug-free. Isn't that right, Mr. Gates?

[2] Since you've read the Homework Instructions, you know what the phrase "describe an algorithm" means. Right?

Practice Problems

These remaining practice problems are entirely for your benefit. Don't turn in solutions—we'll just throw them out—but feel free to ask us about these questions during office hours and review sessions. Think of these as potential exam questions (hint, hint).

1. Recall the standard recursive definition of the Fibonacci numbers: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Prove the following identities for all positive integers n and m.

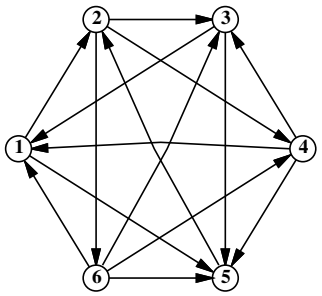
   (a) F_n is even if and only if n is divisible by 3.

   (b) $\sum_{i=0}^{n} F_i = F_{n+2} - 1$

   (c) $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$

   ★(d) If n is an integer multiple of m, then F_n is an integer multiple of F_m.

2. A *tournament* is a directed graph with exactly one edge between every pair of vertices. (Think of the nodes as players in a round-robin tournament, where each edge points from the winner to the loser.) A *Hamiltonian path* is a sequence of directed edges, joined end to end, that visits every vertex exactly once. Prove that every tournament contains at least one Hamiltonian path.



A six-vertex tournament containing the Hamiltonian path $6 \to 4 \to 5 \to 2 \to 3 \to 1$.

3. (a) Prove the following identity by induction:

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}.$$

   (b) Give a non-inductive combinatorial proof of the same identity, by showing that the two sides of the equation count exactly the same thing in two different ways. There is a correct one-sentence proof.

4. (a) Prove that $2^{\lceil \lg n \rceil + \lfloor \lg n \rfloor}/n = \Theta(n)$.

   (b) Is $2^{\lfloor \lg n \rfloor} = \Theta\big(2^{\lceil \lg n \rceil}\big)$? Justify your answer.

   (c) Is $2^{2^{\lfloor \lg \lg n \rfloor}} = \Theta\big(2^{2^{\lceil \lg \lg n \rceil}}\big)$? Justify your answer.

   (d) Prove that if $f(n) = O(g(n))$, then $2^{f(n)} = O(2^{g(n)})$. Justify your answer.

   (e) Prove that $f(n) = O(g(n))$ does *not* imply that $\log(f(n)) = O(\log(g(n)))$?.

   *(f) Prove that $\log^k n = o(n^{1/k})$ for any positive integer k.

5. Solve the following recurrences. State tight asymptotic bounds for each function in the form $\Theta(f(n))$ for some recognizable function $f(n)$. You do not need to turn in proofs (in fact, please *don't* turn in proofs), but you should do them anyway just for practice. If no base cases are given, assume something reasonable (but nontrivial). Extra credit will be given for more exact solutions.

   (a) $A(n) = A(n/2) + n$

   (b) $B(n) = 2B(n/2) + n$

   (c) $C(n) = \min_{0<k<n} \big(C(k) + C(n-k) + 1\big)$, where $C(1) = 1$.

   (d) $D(n) = D(n-1) + 1/n$

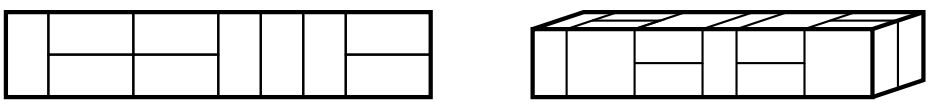
   *(e) $E(n) = 8E(n-1) - 15E(n-2) + 1$

   *(f) $F(n) = (n-1)(F(n-1) + F(n-2))$, where $F(0) = F(1) = 1$

   ★(g) $G(n) = G(n/2) + G(n/4) + G(n/6) + G(n/12) + n$ [Hint: $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = 1$.]

6. (a) A *domino* is a 2×1 or 1×2 rectangle. How many different ways are there to completely fill a $2 \times n$ rectangle with n dominos? Set up a recurrence relation and give an *exact* closed-form solution.

   (b) A *slab* is a three-dimensional box with dimensions $1 \times 2 \times 2$, $2 \times 1 \times 2$, or $2 \times 2 \times 1$. How many different ways are there to fill a $2 \times 2 \times n$ box with n slabs? Set up a recurrence relation and give an *exact* closed-form solution.



A 2×10 rectangle filled with ten dominos, and a $2 \times 2 \times 10$ box filled with ten slabs.

7. Professor George O'Jungle has a favorite 26-node binary tree, whose nodes are labeled by letters of the alphabet. The preorder and postorder sequences of nodes are as follows:

   preorder: M N H C R S K W T G D X I Y A J P O E Z V B U L Q F
   postorder: C W T K S G R H D N A O E P J Y Z I B Q L F U V X M

   Draw Professor O'Jungle's binary tree, and give the inorder sequence of nodes.

8. Alice and Bob each have a fair n-sided die. Alice rolls her die once. Bob then repeatedly throws his die until he rolls a number at least as big as the number Alice rolled. Each time Bob rolls, he pays Alice \$1. (For example, if Alice rolls a 5, and Bob rolls a 4, then a 3, then a 1, then a 5, the game ends and Alice gets \$4. If Alice rolls a 1, then no matter what Bob rolls, the game will end immediately, and Alice will get \$1.)

   Exactly how much money does Alice expect to win at this game? Prove that your answer is correct. If you have to appeal to "intuition" or "common sense", your answer is probably wrong!

9. Prove that for any nonnegative parameters a and b, the following algorithms terminate and produce identical output.

```
SLOWEUCLID(a, b):
  if b > a
      return SLOWEUCLID(b, a)
  else if b = 0
      return a
  else
      return SLOWEUCLID(b, a − b)
```

```
FASTEUCLID(a, b):
  if b = 0
      return a
  else
      return FASTEUCLID(b, a mod b)
```