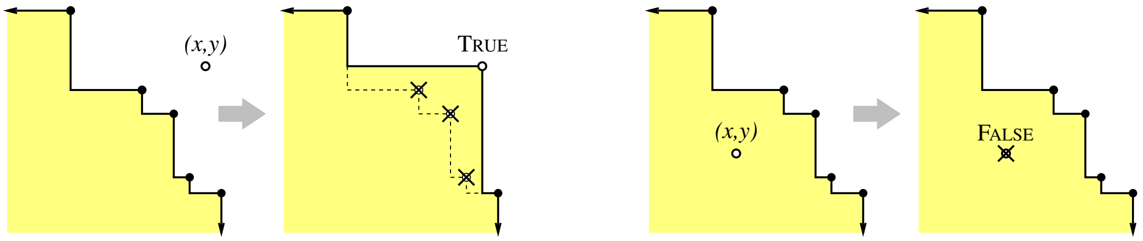
You must turn in this question sheet with your answers.

1. **Déjà vu**

   Prove that any positive integer can be written as the sum of distinct *nonconsecutive* Fibonacci numbers—if F_n appears in the sum, then neither F_{n+1} nor F_{n-1} will. For example: $42 = F_9 + F_6$, $25 = F_8 + F_4 + F_2$, and $17 = F_7 + F_4 + F_2$. *You must give a complete, self-contained proof, not just a reference to the posted homework solutions.*

2. **L'esprit d'escalier**

   Recall that the *staircase* of a set of points consists of the points with no other point both above and to the right. Describe a method to maintain the staircase as new points are added to the set. Specifically, describe and analyze a data structure that stores the staircase of a set of points, and an algorithm INSERT(x, y) that adds the point (x, y) to the set and returns TRUE or FALSE to indicate whether the staircase has changed. Your data structure should use $O(n)$ space, and your INSERT algorithm should run in $O(\log n)$ amortized time.



3. **Engage le jeu que je le gagne**

   A palindrome is a text string that is exactly the same as its reversal, such as DEED, RACECAR, or SAIPPUAKAUPPIAS.[1]

   (a) Describe and analyze an algorithm to find the longest *prefix* of a given string that is also a palindrome. For example, the longest palindrome prefix of ILLINOISURBANACHAMPAIGN is ILLI, and the longest palindrome prefix of HYAKUGOJYUUICHI[2] is the single letter S. For full credit, your algorithm should run in $O(n)$ time.

   (b) Describe and analyze an algorithm to find the length of the longest *subsequence* of a given string that is also a palindrome. For example, the longest palindrome subsequence of ILLINOISURBANACHAMPAIGN is NIAACAAIN (or NIAAHAAIN), and the longest palindrome subsequence of HYAKUGOJYUUICHI is HUUUH[3] (or HUGUH or HUYUH or...). You do not need to compute the actual subsequence; just its length. For full credit, your algorithm should run in $O(n^2)$ time.

[1] Finnish for 'soap dealer'.
[2] Japanese for 'one hundred fifty-one'.
[3] English for 'What the heck are you talking about?'

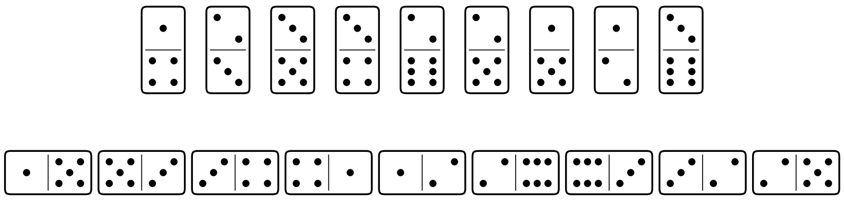
4. **Toute votre base sont appartiennent à nous**

   Prove that *exactly* $2n - 1$ comparisons are required in the worst case to merge two sorted arrays, each with n distinct elements. Describe and analyze an algorithm to prove the upper bound, and use an adversary argument to prove the lower bound. *You must give a complete, self-contained solution, not just a reference to the posted homework solutions.*[4]

5. **Plus ça change, plus ça même chose**

   A domino is a 2×1 rectangle divided into two squares, with a certain number of pips (dots) in each square. In most domino games, the players lay down dominos at either end of a single chain. Adjacent dominos in the chain must have matching numbers. (See the figure below.)

   Describe and analyze an efficient algorithm, or prove that it is NP-hard, to determine whether a given set of n dominos can be lined up in a single chain. For example, for the set of dominos shown below, the correct output is TRUE.



Top: A set of nine dominos
Bottom: The entire set lined up in a single chain

6. **Ceci n'est pas une pipe**

   Consider the following pair of problems:

   - BOXDEPTH: Given a set of n axis-aligned rectangles in the plane and an integer k, decide whether any point in the plane is covered by k or more rectangles.
   - MAXCLIQUE: Given a graph with n vertices and an integer k, decide whether the graph contains a clique with k or more vertices.

   (a) Describe and analyze a reduction of one of these problems to the other.

   (b) MAXCLIQUE is NP-hard. What does your answer to part (a) imply about the complexity of BOXDEPTH?

7. **C'est magique!** *[1-unit graduate students must answer this question.]*

   The recursion fairy's cousin, the reduction genie, shows up one day with a magical gift for you—a box that determines in constant time the size of the largest clique in any given graph. (Recall that a clique is a subgraph where every pair of vertices is joined by an edge.) The magic box does not tell you *where* the largest clique is, only its size. Describe and analyze an algorithm to actually find the largest clique in a given graph **in polynomial time**, using this magic box.

[4]The posted solution for this Homework 5 practice problem was incorrect. So don't use it!