CS 373: Combinatorial Algorithms, Fall 2000

http://www-courses.cs.uiuc.edu/~cs373

Homework 6 (due Thur. December 7, 2000 at 11:59.99 p.m.)

Name:		
Net ID:	Alias:	U $^3\!/_4$ 1

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Starting with Homework 1, homeworks may be done in teams of up to three people. Each team turns in just one solution, and every member of a team gets the same grade. Since 1-unit graduate students are required to solve problems that are worth extra credit for other students, **1-unit grad students may not be on the same team as 3/4-unit grad students or undergraduates.**

Neatly print your name(s), NetID(s), and the alias(es) you used for Homework 0 in the boxes above. Please also tell us whether you are an undergraduate, 3/4-unit grad student, or 1-unit grad student by circling U, $^3\!/_4$, or 1, respectively. Staple this sheet to the top of your homework.

Note: You will be held accountable for the appropriate responses for answers (e.g. give models, proofs, analyses, etc). For NP-complete problems you should prove everything rigorously, i.e. for showing that it is in NP, give a description of a certificate and a poly time algorithm to verify it, and for showing NP-hardness, you must show that your reduction is polytime (by similarly proving something about the algorithm that does the transformation) and proving both directions of the 'if and only if' (a solution of one is a solution of the other) of the many-one reduction.

Required Problems

1. Complexity

   (a) Prove that P $\subseteq$ co-NP.

   (b) Show that if NP $\neq$ co-NP, then *every* NP-complete problem is *not* a member of co-NP.

2. 2-CNF-SAT

   Prove that deciding satisfiability when all clauses have at most 2 literals is in P.

3. 3SUM
   Describe an algorithm that solves the following problem as quickly as possible: Given a set of n numbers, does it contain three elements whose sum is zero? For example, your algorithm should answer TRUE for the set $\{-5, -17, 7, -4, 3, -2, 4\}$, since $-5 + 7 + (-2) = 0$, and FALSE for the set $\{-6, 7, -4, -13, -2, 5, 13\}$.

4. Graph Problems
   (a) SUBGRAPH-ISOMORPHISM
   Show that the problem of deciding whether one graph is a subgraph of another is NP-complete.
   (b) LONGEST-PATH
   Show that the problem of deciding whether an unweighted undirected graph has a path of length greater than k is NP-complete.

5. RECTANGLE-COVER
   (a) Consider the following problem A: given a set of axis-aligned rectangles in a plane, decide whether there is a point in the plane that is covered by k rectangles (or more). Now also consider the CLIQUE problem. Describe and analyze a reduction of one to the other.
   (b) Finding the largest clique in an arbitrary graph is NP-hard. What does this imply about the complexity of finding a point that lies inside the largest number of rectangles?

6. PARTITION, SUBSET-SUM
   [This problem is required only for graduate students taking CS 373 for a full unit; anyone else can submit a solution for extra credit.]
   PARTITION is the problem of deciding, given a set of numbers, whether there exists a subset whose sum equals the sum of the complement, i.e. given $S = s_1, s_2 \ldots, s_n$, does there exist a subset S' such that $\sum_{s \in S'} s = \sum_{t \in S - S'} t$. SUBSET-SUM is the problem of deciding, given a set of numbers and a target sum, whether there exists a subset whose sum equals the target, i.e. given $S = s_1, s_2 \ldots, s_n$ and k, does there exist a subset S' such that $\sum_{s \in S'} s = k$. Give two reduction, one in both directions.

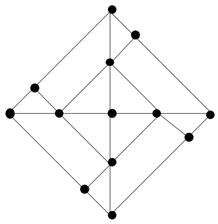

Figure 1. Gadget for PLANAR-3-COLOR.

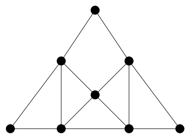

Figure 2. Gadget for DEGREE-4-PLANAR-3-COLOR.

Practice Problems

1. Consider finding the median of 5 numbers by using only comparisons. What is the exact worst case number of comparisons needed to find the median. Justify (exhibit a set that cannot be done in one less comparisons). Do the same for 6 numbers.

2. EXACT-COVER-BY-4-SETS
   The EXACT-COVER-BY-3-SETS problem is defines as the following: given a finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X, does C contain an *exact cover* for X, that is, a subcollection $C' \subseteq C$ such that every element of X occurs in exactly one member of C'?

   Given that EXACT-COVER-BY-3-SETS is NP-complete, show that EXACT-COVER-BY-4-SETS is also NP-complete.

3. PLANAR-3-COLOR
   Using 3-COLOR, and the 'gadget' in figure 3, prove that the problem of deciding whether a planar graph can be 3-colored is NP-complete. Hint: show that the gadget can be 3-colored, and then replace any crossings in a planar embedding with the gadget appropriately.

4. DEGREE-4-PLANAR-3-COLOR
   Using the previous result, and the 'gadget' in figure 4, prove that the problem of deciding whether a planar graph with no vertex of degree greater than four can be 3-colored is NP-complete. Hint: show that you can replace any vertex with degree greater than 4 with a collection of gadgets connected in such a way that no degree is greater than four.

5. Poly time subroutines can lead to exponential algorithms
   Show that an algorithm that makes at most a constant number of calls to polynomial-time subroutines runs in polynomial time, but that a polynomial number of calls to polynomial-time subroutines may result in an exponential-time algorithm.

6. (a) Prove that if G is an undirected bipartite graph with an odd number of vertices, then G is nonhamiltonian. Give a polynomial time algorithm algorithm for finding a **hamiltonian cycle** in an undirected bipartite graph or establishing that it does not exist.
   (b) Show that the **hamiltonian-path** problem can be solved in polynomial time on directed acyclic graphs by giving an efficient algorithm for the problem.
   (c) Explain why the results in previous questions do not contradict the facts that both HAM-CYCLE and HAM-PATH are NP-complete problems.

7. Consider the following pairs of problems:
   (a) MIN SPANNING TREE and MAX SPANNING TREE
   (b) SHORTEST PATH and LONGEST PATH
   (c) TRAVELING SALESMAN PROBLEM and VACATION TOUR PROBLEM (the longest tour is sought).
   (d) MIN CUT and MAX CUT (between s and t)
   (e) EDGE COVER and VERTEX COVER
   (f) TRANSITIVE REDUCTION and MIN EQUIVALENT DIGRAPH

   (all of these seem dual or opposites, except the last, which are just two versions of minimal representation of a graph).

   Which of these pairs are polytime equivalent and which are not? Why?

★8. GRAPH-ISOMORPHISM
   Consider the problem of deciding whether one graph is isomorphic to another.
   (a) Give a brute force algorithm to decide this.
   (b) Give a dynamic programming algorithm to decide this.
   (c) Give an efficient probabilistic algorithm to decide this.
   (d) Either prove that this problem is NP-complete, give a poly time algorithm for it, or prove that neither case occurs.

9. Prove that PRIMALITY (Given n, is n prime?) is in NP $\cap$ co-NP. Hint: co-NP is easy (what's a certificate for showing that a number is composite?). For NP, consider a certificate involving primitive roots and recursively their primitive roots. Show that knowing this tree of primitive roots can be checked to be correct and used to show that n is prime, and that this check takes poly time.

10. How much wood would a woodchuck chuck if a woodchuck could chuck wood?

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5. BIN-PACKING Consider the bin-packing problem: given a finite set U of n items and the positive integer size $s(u)$ of each item $u \in U$, can U be partitioned into k disjoint sets $U_1, \ldots, U_k$ such that the sum of the sizes of the items in each set does not exceed B? Show that the bin-packing problem is NP-Complete. [Hint: Use the result from the previous problem.]

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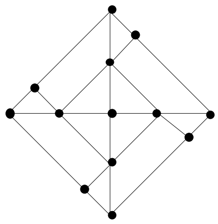

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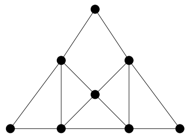

Figure 2. Gadget for DEGREE-4-PLANAR-3-COLOR.

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