CS 373: Combinatorial Algorithms, Spring 1999

http://www-courses.cs.uiuc.edu/ cs373

Homework 0 (due January 26, 1999 by the beginning of class)

Name:	
Net ID:	Alias:

Neatly print your name (first name first, with no comma), your network ID, and a short alias into the boxes above. Do not *sign* your name. Do not write your Social Security number. Staple this sheet of paper to the top of your homework. Grades will be listed on the course web site by alias, so your alias should not resemble your name (or your Net ID). If you do not give yourself an alias, you will be stuck with one we give you, no matter how much you hate it.

Everyone must do the problems marked ▶. Problems marked ▷ are for 1-unit grad students and others who want extra credit. (There's no such thing as "partial extra credit"!) Unmarked problems are extra practice problems for your benefit, which will not be graded. Think of them as potential exam questions.

Hard problems are marked with a star; the bigger the star, the harder the problem.

This homework tests your familiarity with the prerequisite material from CS 225 and CS 273 (and *their* prerequisites)—many of these problems appeared on homeworks and/or exams in those classes—primarily to help you identify gaps in your knowledge. **You are responsible for filling those gaps on your own.**

Undergrad/.75U Grad/1U Grad Problems

▶1. **[173/273]**

(a) Prove that any positive integer can be written as the sum of distinct powers of 2. (For example: $42 = 2^5 + 2^3 + 2^1$, $25 = 2^4 + 2^3 + 2^0$, $17 = 2^4 + 2^0$.)

(b) Prove that any positive integer can be written as the sum of distinct *nonconsecutive* Fibonacci numbers—if F_n appears in the sum, then neither F_{n+1} nor F_{n-1} will. (For example: $42 = F_9 + F_6$, $25 = F_8 + F_4 + F_2$, $17 = F_7 + F_4 + F_2$.)

(c) Prove that *any* integer can be written in the form $\sum_i \pm 3^i$, where the exponents i are distinct non-negative integers. (For example: $42 = 3^4 - 3^3 - 3^2 - 3^1$, $25 = 3^3 - 3^1 + 3^0$, $17 = 3^3 - 3^2 - 3^0$.)

▶2. **[225/273]** Sort the following functions from asymptotically smallest to largest, indicating ties if there are any: n, $\lg n$, $\lg\lg^* n$, $\lg^* \lg n$, $\lg^* n$, $n \lg n$, $\lg(n \lg n)$, $n^{n/\lg n}$, $n^{\lg n}$, $(\lg n)^n$, $(\lg n)^{\lg n}$, $2^{\sqrt{\lg n \lg\lg n}}$, 2^n, $n^{\lg\lg n}$, $\sqrt[1000]{n}$, $(1 + \frac{1}{1000})^n$, $(1 - \frac{1}{1000})^n$, $\lg^{1000} n$, $\lg^{(1000)} n$, $\log_{1000} n$, $\lg^n 1000$, 1.

[To simplify notation, write $f(n) \ll g(n)$ to mean $f(n) = o(g(n))$ and $f(n) \equiv g(n)$ to mean $f(n) = \Theta(g(n))$. For example, the functions n^2, n, $\binom{n}{2}$, n^3 could be sorted as follows: $n \ll n^2 \equiv \binom{n}{2} \ll n^3$.]

3. **[273/225]** Solve the following recurrences. State tight asymptotic bounds for each function in the form $\Theta(f(n))$ for some recognizable function $f(n)$. You do not need to turn in proofs (in fact, please *don't* turn in proofs), but you should do them anyway just for practice. Assume reasonable (nontrivial) base cases. Extra credit will be given for more exact solutions.

   ▶(a) $A(n) = A(n/2) + n$

   (b) $B(n) = 2B(n/2) + n$

   ▶(c) $C(n) = 3C(n/2) + n$

   (d) $D(n) = \max\limits_{n/3<k<2n/3} \big(D(k) + D(n-k) + n\big)$

   ▶(e) $E(n) = \min\limits_{0<k<n} \big(E(k) + E(n-k) + 1\big)$

   (f) $F(n) = 4F(\lfloor n/2 \rfloor + 5) + n$

   ▶(g) $G(n) = G(n-1) + 1/n$

   *(h) $H(n) = H(n/2) + H(n/4) + H(n/6) + H(n/12) + n$ [Hint: $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = 1$.]

   ▶*(i) $I(n) = 2I(n/2) + n/\lg n$

   *(j) $J(n) = \dfrac{J(n-1)}{J(n-2)}$

▶4. **[273]** Alice and Bob each have a fair n-sided die. Alice rolls her die once. Bob then repeatedly throws his die until the number he rolls is at least as big as the number Alice rolled. Each time Bob rolls, he pays Alice \$1. (For example, if Alice rolls a 5, and Bob rolls a 4, then a 3, then a 1, then a 5, the game ends and Alice gets \$4. If Alice rolls a 1, then no matter what Bob rolls, the game will end immediately, and Alice will get \$1.)

   Exactly how much money does Alice expect to win at this game? Prove that your answer is correct. (If you have to appeal to "intuition" or "common sense", your answer is probably wrong.)

▶5. **[225]** George has a 26-node binary tree, with each node labeled by a unique letter of the alphabet. The preorder and postorder sequences of nodes are as follows:

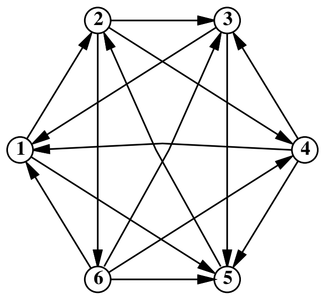
   preorder: `M N H C R S K W T G D X I Y A J P O E Z V B U L Q F`
   postorder: `C W T K S G R H D N A O E P J Y Z I B Q L F U V X M`

   Draw George's binary tree.

Only 1U Grad Problems

▷*1. **[225/273]** A *tournament* is a directed graph with exactly one edge between every pair of vertices. (Think of the nodes as players in a round-robin tournament, where each edge points from the winner to the loser.) A *Hamiltonian path* is a sequence of directed edges, joined end to end, that visits every vertex exactly once.

   Prove that every tournament contains at least one Hamiltonian path.



A six-vertex tournament containing the Hamiltonian path $6 \to 4 \to 5 \to 2 \to 3 \to 1$.

Practice Problems

1. **[173/273]** Recall the standard recursive definition of the Fibonacci numbers: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Prove the following identities for all positive integers n and m.

   (a) F_n is even if and only if n is divisible by 3.

   (b) $\sum_{i=0}^{n} F_i = F_{n+2} - 1$

   (c) $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$

   ★(d) If n is an integer multiple of m, then F_n is an integer multiple of F_m.

2. **[225/273]**

   (a) Prove that $2^{\lceil \lg n \rceil + \lfloor \lg n \rfloor}/n = \Theta(n)$.

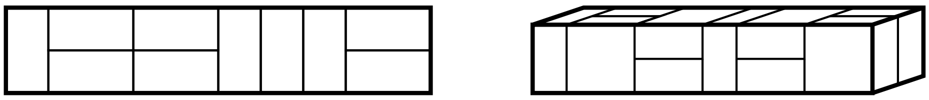
   (b) Is $2^{\lfloor \lg n \rfloor} = \Theta\big(2^{\lceil \lg n \rceil}\big)$? Justify your answer.

   (c) Is $2^{2^{\lfloor \lg \lg n \rfloor}} = \Theta\big(2^{2^{\lceil \lg \lg n \rceil}}\big)$? Justify your answer.

3. **[273]**

   (a) A *domino* is a 2×1 or 1×2 rectangle. How many different ways are there to completely fill a $2 \times n$ rectangle with n dominos?

   (b) A *slab* is a three-dimensional box with dimensions $1 \times 2 \times 2$, $2 \times 1 \times 2$, or $2 \times 2 \times 1$. How many different ways are there to fill a $2 \times 2 \times n$ box with n slabs? Set up a recurrence relation and give an *exact* closed-form solution.



A 2×10 rectangle filled with ten dominos, and a $2 \times 2 \times 10$ box filled with ten slabs.

4. **[273]** Penn and Teller have a special deck of fifty-two cards, with no face cards and nothing but clubs—the ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ..., 52 of clubs. (They're big cards.) Penn shuffles the deck until each each of the 52! possible orderings of the cards is equally likely. He then takes cards one at a time from the top of the deck and gives them to Teller, stopping as soon as he gives Teller the three of clubs.

(a) On average, how many cards does Penn give Teller?

(b) On average, what is the smallest-numbered card that Penn gives Teller?

*(c) On average, what is the largest-numbered card that Penn gives Teller?

Prove that your answers are correct. (If you have to appeal to "intuition" or "common sense", your answers are probably wrong.) [Hint: Solve for an n-card deck, and then set n to 52.]

5. **[273/225]** Prove that for any nonnegative parameters a and b, the following algorithms terminate and produce identical output.

```
SlowEuclid(a, b):
    if b > a
        return SlowEuclid(b, a)
    else if b == 0
        return a
    else
        return SlowEuclid(a, b − a)
```

```
FastEuclid(a, b):
    if b == 0
        return a
    else
        return FastEuclid(b, a mod b)
```