4 Euler Tours and Hamilton Cycles

4.1 EULER TOURS

A trail that traverses every edge of G is called an *Euler trail* of G because Euler was the first to investigate the existence of such trails in graphs. In the earliest known paper on graph theory (Euler, 1736), he showed that it was impossible to cross each of the seven bridges of Königsberg once and only once during a walk through the town. A plan of Königsberg and the river Pregel is shown in figure 4.1a. As can be seen, proving that such a walk is impossible amounts to showing that the graph of figure 4.1b contains no Euler trail.

A *tour* of G is a closed walk that traverses each edge of G at least once. An *Euler tour* is a tour which traverses each edge exactly once (in other words, a closed Euler trail). A graph is *eulerian* if it contains an Euler tour.

Theorem 4.1 A nonempty connected graph is eulerian if and only if it has no vertices of odd degree.

Proof Let G be eulerian, and let C be an Euler tour of G with origin (and terminus) u. Each time a vertex v occurs as an internal vertex of C, two of the edges incident with v are accounted for. Since an Euler tour contains

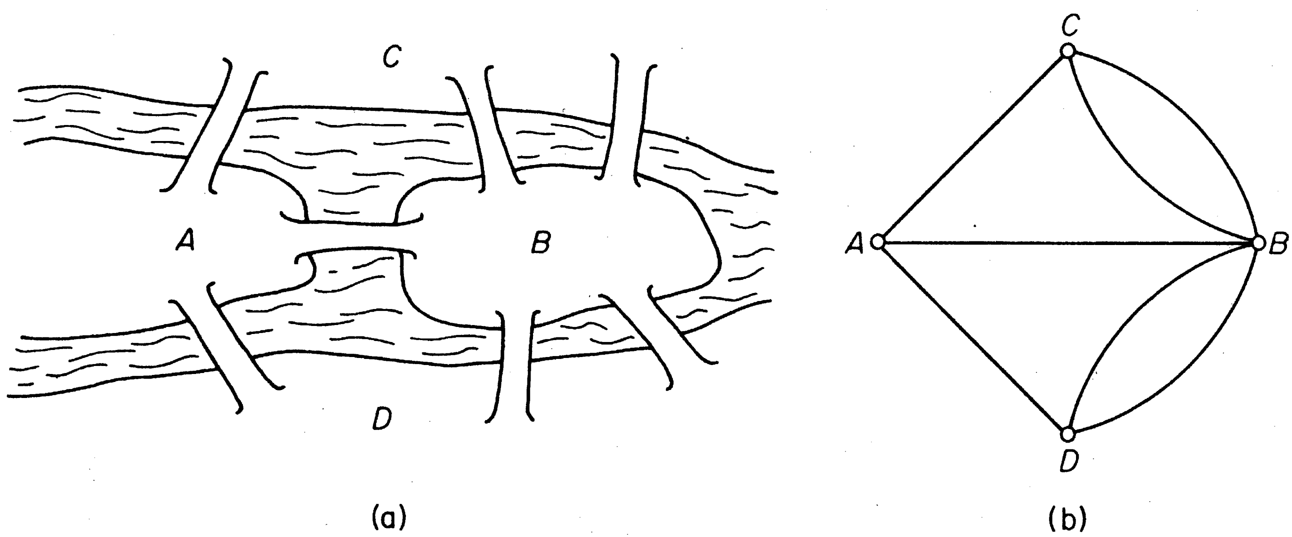


Figure 4.1. The bridges of Königsberg and their graph

every edge of G, $d(v)$ is even for all $v \neq u$. Similarly, since C starts and ends at u, $d(u)$ is also even. Thus G has no vertices of odd degree.

Conversely, suppose that G is a noneulerian connected graph with at least one edge and no vertices of odd degree. Choose such a graph G with as few edges as possible. Since each vertex of G has degree at least two, G contains a closed trail (exercise 1.7.2). Let C be a closed trail of maximum possible length in G. By assumption, C is not an Euler tour of G and so $G-E(C)$ has some component G' with $\varepsilon(G')>0$. Since C is itself eulerian, it has no vertices of odd degree; thus the connected graph G' also has no vertices of odd degree. Since $\varepsilon(G')<\varepsilon(G)$, it follows from the choice of G that G' has an Euler tour C'. Now, because G is connected, there is a vertex v in $V(C) \cap V(C')$, and we may assume, without loss of generality, that v is the origin and terminus of both C and C'. But then CC' is a closed trail of G with $\varepsilon(CC')>\varepsilon(C)$, contradicting the choice of C □

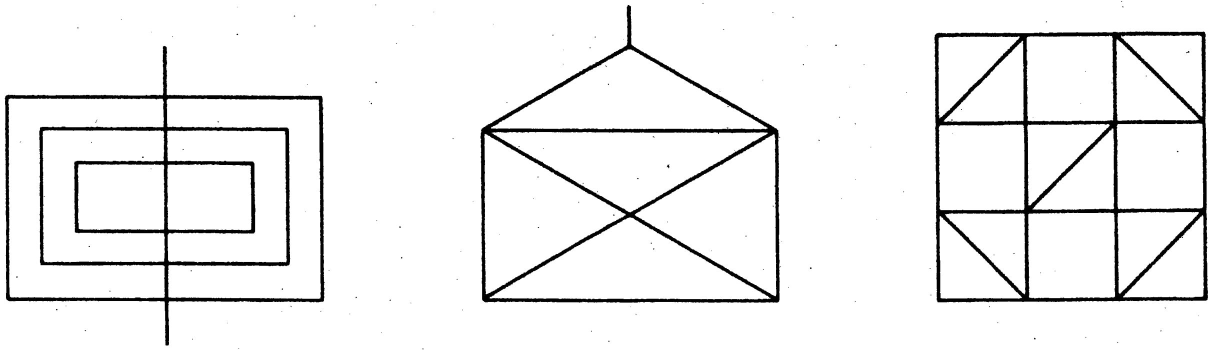
Corollary 4.1 A connected graph has an Euler trail if and only if it has at most two vertices of odd degree.

Proof If G has an Euler trail then, as in the proof of theorem 4.1, each vertex other than the origin and terminus of this trail has even degree.

Conversely, suppose that G is a nontrivial connected graph with at most two vertices of odd degree. If G has no such vertices then, by theorem 4.1, G has a closed Euler trail. Otherwise, G has exactly two vertices, u and v, of odd degree. In this case, let $G+e$ denote the graph obtained from G by the addition of a new edge e joining u and v. Clearly, each vertex of $G+e$ has even degree and so, by theorem 4.1, $G+e$ has an Euler tour $C = v_0e_1v_1 \ldots e_{\varepsilon+1}v_{\varepsilon+1}$, where $e_1 = e$. The trail $v_1e_2v_2 \ldots e_{\varepsilon+1}v_{\varepsilon+1}$ is an Euler trail of G □

Exercises

4.1.1 Which of the following figures can be drawn without lifting one's pen from the paper or covering a line more than once?



4.1.2 If possible, draw an eulerian graph G with ν even and ε odd; otherwise, explain why there is no such graph.

4.1.3 Show that if G is eulerian, then every block of G is eulerian.

4.1.4 Show that if G has no vertices of odd degree, then there are edge-disjoint cycles $C_1, C_2, \ldots, C_m$ such that $E(G) = E(C_1) \cup E(C_2) \cup \ldots \cup E(C_m)$.

4.1.5 Show that if a connected graph G has $2k > 0$ vertices of odd degree, then there are k edge-disjoint trails $Q_1, Q_2, \ldots, Q_k$ in G such that $E(G) = E(Q_1) \cup E(Q_2) \cup \ldots \cup E(Q_k)$.

4.1.6* Let G be nontrivial and eulerian, and let $v \in V$. Show that every trail of G with origin v can be extended to an Euler tour of G if and only if $G - v$ is a forest. (O. Ore)

4.2 HAMILTON CYCLES

A path that contains every vertex of G is called a *Hamilton path* of G; similarly, a *Hamilton cycle* of G is a cycle that contains every vertex of G. Such paths and cycles are named after Hamilton (1856), who described, in a letter to his friend Graves, a mathematical game on the dodecahedron (figure 4.2a) in which one person sticks five pins in any five consecutive vertices and the other is required to complete the path so formed to a

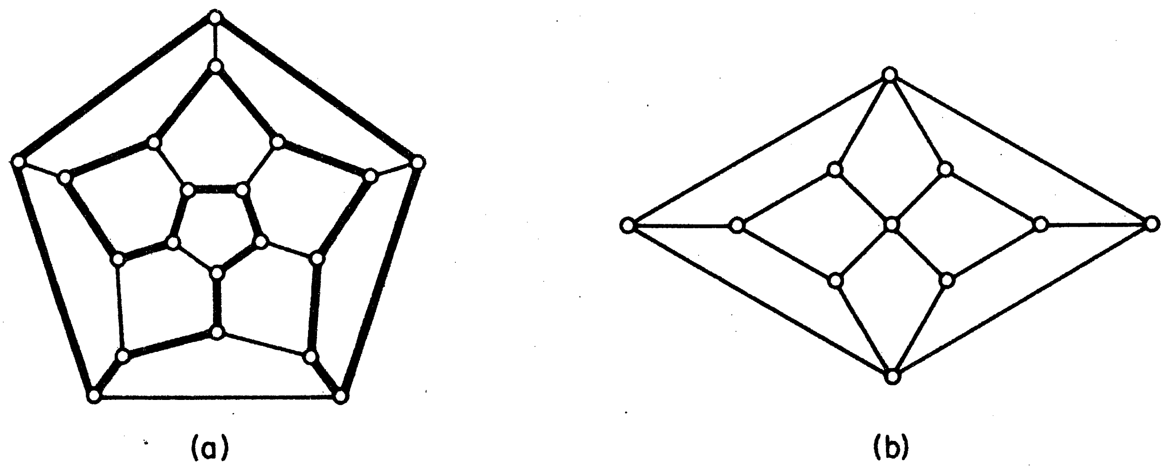


Figure 4.2. (a) The dodecahedron; (b) the Herschel graph

spanning cycle. A graph is *hamiltonian* if it contains a Hamilton cycle. The dodecahedron is hamiltonian (see figure 4.2a); the Herschel graph (figure 4.2b) is nonhamiltonian, because it is bipartite and has an odd number of vertices.

In contrast with the case of eulerian graphs, no nontrivial necessary and sufficient condition for a graph to be hamiltonian is known; in fact, the problem of finding such a condition is one of the main unsolved problems of graph theory.

We shall first present a simple, but useful, necessary condition.

Theorem 4.2 If G is hamiltonian then, for every nonempty proper subset S of V

$$\omega(G-S) \leq |S| \tag{4.1}$$

Proof Let C be a Hamilton cycle of G. Then, for every nonempty proper subset S of V

$$\omega(C-S) \leq |S|$$

Also, $C-S$ is a spanning subgraph of $G-S$ and so

$$\omega(G-S) \leq \omega(C-S)$$

The theorem follows □

As an illustration of the above theorem, consider the graph of figure 4.3. This graph has nine vertices; on deleting the three indicated in black, four components remain. Therefore (4.1) is not satisfied and it follows from theorem 4.2 that the graph is nonhamiltonian.

We thus see that theorem 4.2 can sometimes be applied to show that a particular graph is nonhamiltonian. However, this method does not always

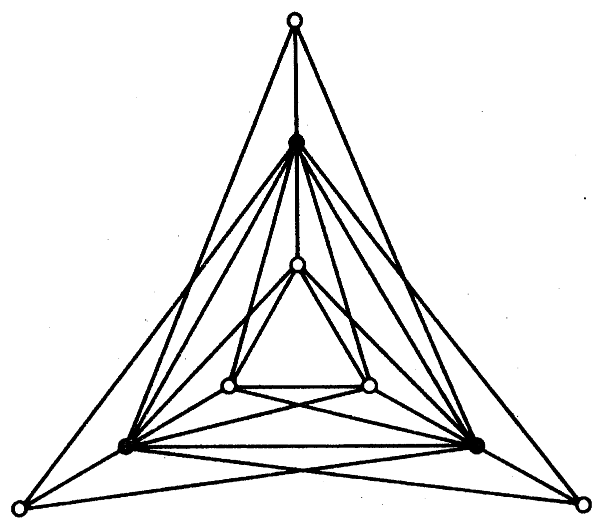


Figure 4.3

work; for instance, the Petersen graph (figure 4.4) is nonhamiltonian, but one cannot deduce this by using theorem 4.2.

We now discuss sufficient conditions for a graph G to be hamiltonian; since a graph is hamiltonian if and only if its underlying simple graph is hamiltonian, it suffices to limit our discussion to simple graphs. We start with a result due to Dirac (1952).

Theorem 4.3 If G is a simple graph with $\nu \geq 3$ and $\delta \geq \nu/2$, then G is hamiltonian.

Proof By contradiction. Suppose that the theorem is false, and let G be a maximal nonhamiltonian simple graph with $\nu \geq 3$ and $\delta \geq \nu/2$. Since $\nu \geq 3$, G cannot be complete. Let u and v be nonadjacent vertices in G. By the choice of G, $G+uv$ is hamiltonian. Moreover, since G is nonhamiltonian,

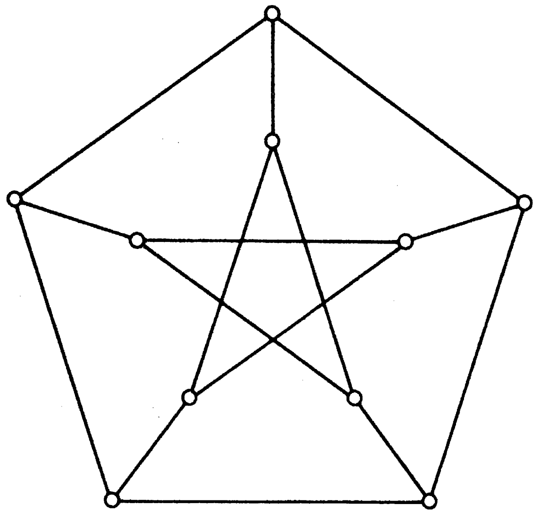


Figure 4.4. The Petersen graph

each Hamilton cycle of $G+uv$ must contain the edge uv. Thus there is a Hamilton path $v_1v_2 \ldots v_\nu$ in G with origin $u=v_1$ and terminus $v=v_\nu$. Set

$$S=\{v_i \mid uv_{i+1} \in E\} \quad \text{and} \quad T=\{v_i \mid v_iv \in E\}$$

Since $v_\nu \notin S \cup T$ we have

$$|S \cup T| < \nu \tag{4.2}$$

Furthermore

$$|S \cap T| = 0 \tag{4.3}$$

since if $S \cap T$ contained some vertex v_i, then G would have the Hamilton cycle $v_1v_2 \ldots v_iv_\nu v_{\nu-1} \ldots v_{i+1}v_1$, contrary to assumption (see figure 4.5).

Using (4.2) and (4.3) we obtain

$$d(u)+d(v)=|S|+|T|=|S \cup T|+|S \cap T| < \nu \tag{4.4}$$

But this contradicts the hypothesis that $\delta \geq \nu/2$ □

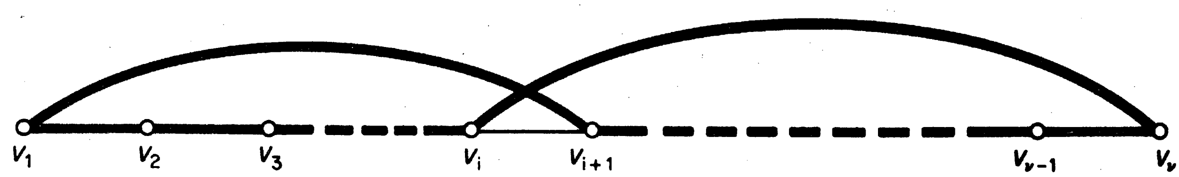


Figure 4.5

Bondy and Chvátal (1974) observed that the proof of theorem 4.3 can be modified to yield stronger sufficient conditions than that obtained by Dirac. The basis of their approach is the following lemma.

Lemma 4.4.1 Let G be a simple graph and let u and v be nonadjacent vertices in G such that

$$d(u)+d(v) \geq \nu \tag{4.5}$$

Then G is hamiltonian if and only if $G+uv$ is hamiltonian.

Proof If G is hamiltonian then, trivially, so too is $G+uv$. Conversely, suppose that $G+uv$ is hamiltonian but G is not. Then, as in the proof of theorem 4.3, we obtain (4.4). But this contradicts hypothesis (4.5) □

Lemma 4.4.1 motivates the following definition. The *closure* of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least ν until no such pair remains. We denote the closure of G by $c(G)$.

Lemma 4.4.2 $c(G)$ is well defined.

Proof Let G_1 and G_2 be two graphs obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least ν until no such pair remains. Denote by $e_1, e_2, \ldots, e_m$ and $f_1, f_2, \ldots, f_n$ the sequences of edges added to G in obtaining G_1 and G_2, respectively. We shall show that each e_i is an edge of G_2 and each f_j is an edge of G_1.

If possible, let $e_{k+1}=uv$ be the first edge in the sequence $e_1, e_2, \ldots, e_n$ that is not an edge of G_2. Set $H=G+\{e_1, e_2, \ldots, e_k\}$. It follows from the definition of G_1 that

$$d_H(u)+d_H(v)\geq \nu$$

By the choice of e_{k+1}, H is a subgraph of G_2. Therefore

$$d_{G_2}(u)+d_{G_2}(v)\geq \nu$$

This is a contradiction, since u and v are nonadjacent in G_2. Therefore each e_i is an edge of G_2 and, similarly, each f_j is an edge of G_1. Hence $G_1=G_2$, and $c(G)$ is well defined □

Figure 4.6 illustrates the construction of the closure of a graph G on six vertices. It so happens that in this example $c(G)$ is complete; note, however, that this is by no means always the case.

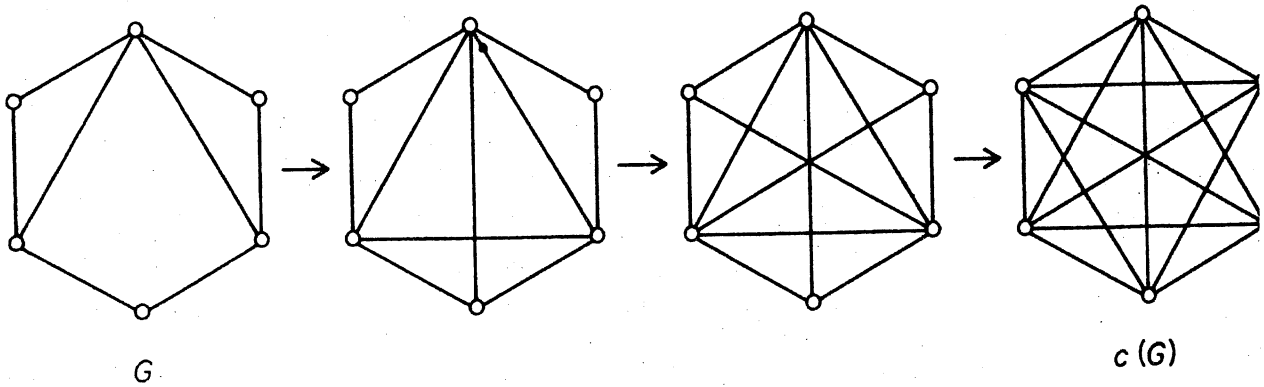


Figure 4.6. The closure of a graph

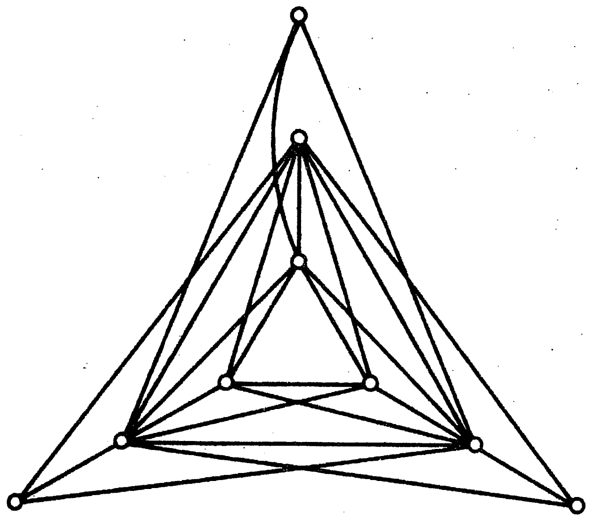


Figure 4.7. A hamiltonian graph

Theorem 4.4 A simple graph is hamiltonian if and only if its closure is hamiltonian.

Proof Apply lemma 4.4.1 each time an edge is added in the formation of the closure □

Theorem 4.4 has a number of interesting consequences. First, upon making the trivial observation that all complete graphs on at least three vertices are hamiltonian, we obtain the following result.

Corollary 4.4 Let G be a simple graph with $\nu \geq 3$. If $c(G)$ is complete, then G is hamiltonian.

Consider, for example, the graph of figure 4.7. One readily checks that its closure is complete. Therefore, by corollary 4.4, it is hamiltonian. It is perhaps interesting to note that the graph of figure 4.7 can be obtained from the graph of figure 4.3 by altering just one end of one edge, and yet we have results (corollary 4.4 and theorem 4.2) which tell us that this one is hamiltonian whereas the other is not.

Corollary 4.4 can be used to deduce various sufficient conditions for a graph to be hamiltonian in terms of its vertex degrees. For example, since $c(G)$ is clearly complete when $\delta \geq \nu/2$, Dirac's condition (theorem 4.3) is an immediate corollary. A more general condition than that of Dirac was obtained by Chvátal (1972).

Theorem 4.5 Let G be a simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1 \leq d_2 \leq \ldots \leq d_\nu$ and $\nu \geq 3$. Suppose that there is no value of m less than $\nu/2$ for which $d_m \leq m$ and $d_{\nu-m} < \nu - m$. Then G is hamiltonian.

Proof Let G satisfy the hypothesis of the theorem. We shall show that its closure $c(G)$ is complete, and the conclusion will then follow from corollary 4.4. We denote the degree of a vertex v in $c(G)$ by $d'(v)$.

Assume that $c(G)$ is not complete, and let u and v be two nonadjacent vertices in $c(G)$ with

$$d'(u) \le d'(v) \tag{4.6}$$

and $d'(u)+d'(v)$ as large as possible; since no two nonadjacent vertices in $c(G)$ can have degree sum ν or more, we have

$$d'(u)+d'(v) < \nu \tag{4.7}$$

Now denote by S the set of vertices in $V\backslash\{v\}$ which are nonadjacent to v in $c(G)$, and by T the set of vertices in $V\backslash\{u\}$ which are nonadjacent to u in $c(G)$. Clearly

$$|S| = \nu - 1 - d'(v) \quad \text{and} \quad |T| = \nu - 1 - d'(u) \tag{4.8}$$

Furthermore, by the choice of u and v, each vertex in S has degree at most $d'(u)$ and each vertex in $T\cup\{u\}$ has degree at most $d'(v)$. Setting $d'(u)=m$ and using (4.7) and (4.8), we find that $c(G)$ has at least m vertices of degree at most m and at least $\nu-m$ vertices of degree less than $\nu-m$. Because G is a spanning subgraph of $c(G)$, the same is true of G; therefore $d_m \le m$ and $d_{\nu-m} < \nu - m$. But this is contrary to hypothesis since, by (4.6) and (4.7), $m < \nu/2$. We conclude that $c(G)$ is indeed complete and hence, by corollary 4.4, that G is hamiltonian □

One can often deduce that a given graph is hamiltonian simply by computing its degree sequence and applying theorem 4.5. This method works with the graph of figure 4.7 but not with the graph G of figure 4.6, even though the closure of the latter graph is complete. From these examples, we see that theorem 4.5 is stronger than theorem 4.3 but not as strong as corollary 4.4.

A sequence of real numbers $(p_1, p_2, \ldots, p_n)$ is said to be *majorised* by another such sequence $(q_1, q_2, \ldots, q_n)$ if $p_i \le q_i$ for $1 \le i \le n$. A graph G is *degree-majorised* by a graph H if $\nu(G)=\nu(H)$ and the nondecreasing degree sequence of G is majorised by that of H. For instance, the 5-cycle is degree-majorised by $K_{2,3}$ because (2, 2, 2, 2, 2) is majorised by (2, 2, 2, 3, 3). The family of degree-maximal nonhamiltonian graphs (those that are degree-majorised by no others) admits of a simple description. We first introduce the notion of the join of two graphs. The *join* $G \vee H$ of disjoint graphs G and H is the graph obtained from $G+H$ by joining each vertex of G to each vertex of H; it is represented diagrammatically as in figure 4.8.

Now, for $1 \le m < n/2$, let $C_{m,n}$ denote the graph $K_m \vee (K_m^c + K_{n-2m})$, depicted in figure 4.9*a*; two specific examples, $C_{1,5}$ and $C_{2,5}$, are shown in figures 4.9*b* and 4.9*c*.

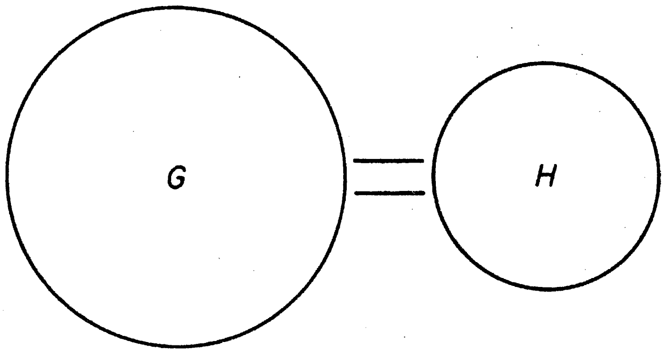


Figure 4.8. The join of G and H

That $C_{m,n}$ is nonhamiltonian follows immediately from theorem 4.2; for if S denotes the set of m vertices of degree $n-1$ in $C_{m,n}$, we have $\omega(C_{m,n}-S)=m+1>|S|$.

Theorem 4.6 (Chvátal, 1972) If G is a nonhamiltonian simple graph with $\nu \geq 3$, then G is degree-majorised by some $C_{m,\nu}$.

Proof Let G be a nonhamiltonian simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1 \leq d_2 \leq \ldots \leq d_\nu$ and $\nu \geq 3$. Then, by theorem 4.5, there exists $m<\nu/2$ such that $d_m \leq m$ and $d_{\nu-m}<\nu-m$. Therefore $(d_1, d_2, \ldots, d_\nu)$ is majorised by the sequence

$$(m, \ldots, m, \nu-m-1, \ldots, \nu-m-1, \nu-1, \ldots, \nu-1)$$

with m terms equal to m, $\nu-2m$ terms equal to $\nu-m-1$ and m terms equal to $\nu-1$, and this latter sequence is the degree sequence of $C_{m,\nu}$ □

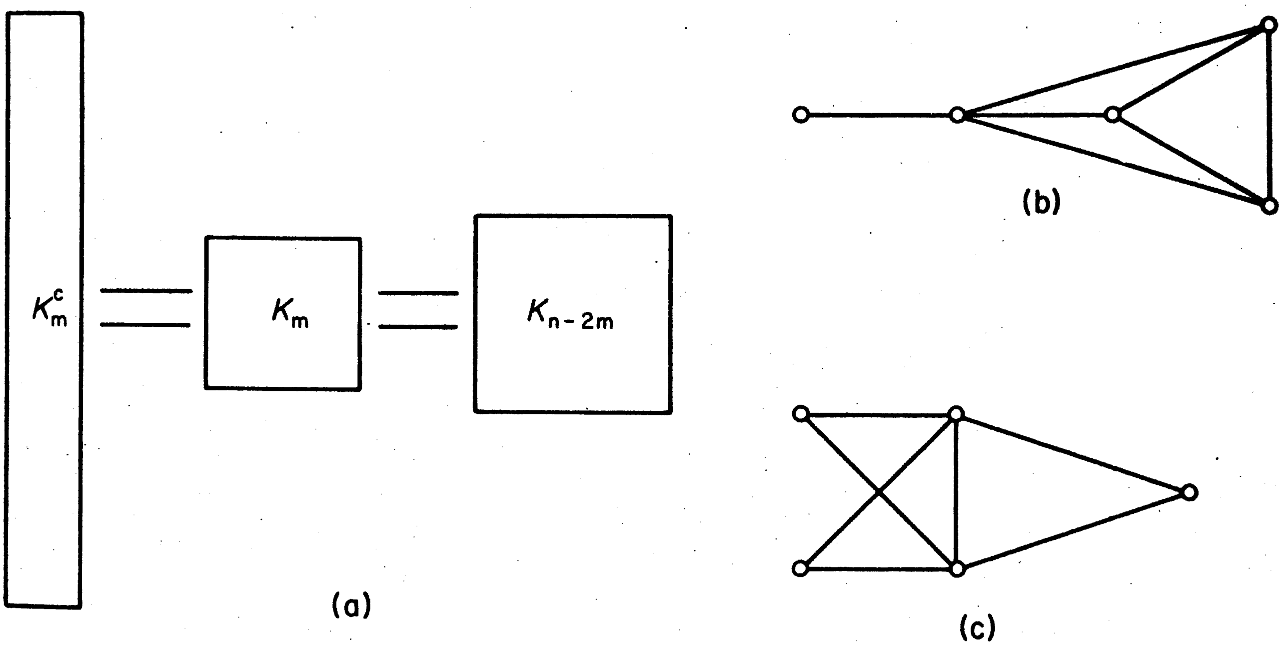


Figure 4.9. (*a*) $C_{m,n}$; (*b*) $C_{1,5}$; (*c*) $C_{2,5}$

From theorem 4.6 we can deduce a result due to Ore (1961) and Bondy (1972).

Corollary 4.6 If G is a simple graph with $\nu \geq 3$ and $\varepsilon > \binom{\nu-1}{2}+1$, then G is hamiltonian. Moreover, the only nonhamiltonian simple graphs with ν vertices and $\binom{\nu-1}{2}+1$ edges are $C_{1,\nu}$ and, for $\nu=5$, $C_{2,5}$.

Proof Let G be a nonhamiltonian simple graph with $\nu \geq 3$. By theorem 4.6, G is degree-majorised by $C_{m,\nu}$ for some positive integer $m < \nu/2$. Therefore, by theorem 1.1,

$$\begin{aligned}\varepsilon(G) &\leq \varepsilon(C_{m,\nu}) & (4.9)\\ &= \tfrac{1}{2}(m^2+(\nu-2m)(\nu-m-1)+m(\nu-1))\\ &= \binom{\nu-1}{2}+1-\tfrac{1}{2}(m-1)(m-2)-(m-1)(\nu-2m-1)\\ &\leq \binom{\nu-1}{2}+1 & (4.10)\end{aligned}$$

Furthermore, equality can only hold in (4.9) if G has the same degree sequence as $C_{m,\nu}$; and equality can only hold in (4.10) if either $m=2$ and $\nu=5$, or $m=1$. Hence $\varepsilon(G)$ can equal $\binom{\nu-1}{2}+1$ only if G has the same degree sequence as $C_{1,\nu}$ or $C_{2,5}$, which is easily seen to imply that $G \cong C_{1,\nu}$ or $G \cong C_{2,5}$ □

Exercises

4.2.1 Show that if either
(*a*) G is not 2-connected, or
(*b*) G is bipartite with bipartition (X, Y) where $|X| \neq |Y|$,
then G is nonhamiltonian.

4.2.2 A mouse eats his way through a $3\times3\times3$ cube of cheese by tunnelling through all of the 27 $1\times1\times1$ subcubes. If he starts at one corner and always moves on to an uneaten subcube, can he finish at the centre of the cube?

4.2.3 Show that if G has a Hamilton path then, for every proper subset S of V, $\omega(G-S) \leq |S|+1$.

4.2.4* Let G be a nontrivial simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1 \leq d_2 \leq \ldots \leq d_\nu$. Show that if there is no value of m less than $(\nu+1)/2$ for which $d_m < m$ and $d_{\nu-m+1} < \nu - m$, then G has a Hamilton path. (V. Chvátal)

4.2.5 (*a*) Let G be a simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$ and let G^c have degree sequence $(d'_1, d'_2, \ldots, d'_\nu)$ where $d_1 \le d_2 \le \ldots \le d_\nu$ and $d'_1 \le d'_2 \le \ldots \le d'_\nu$. Show that if $d_m \ge d'_m$ for all $m \le \nu/2$, then G has a Hamilton path.

(*b*) Deduce that if G is self-complementary, then G has a Hamilton path. (C. R. J. Clapham)

4.2.6* Let G be a simple bipartite graph with bipartition (X, Y), where $|X| = |Y| \ge 2$, and let G have degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1 \le d_2 \le \ldots \le d_\nu$. Show that if there is no value of m less than or equal to $\nu/4$ for which $d_m \le m$ and $d_{\nu/2} \le \nu/2 - m$, then G is hamiltonian. (V. Chvátal)

4.2.7 Prove corollary 4.6 directly from corollary 4.4.

4.2.8 Show that if G is simple with $\nu \ge 6\delta$ and $\varepsilon > \binom{\nu-\delta}{2} + \delta^2$, then G is hamiltonian. (P. Erdös)

4.2.9* Show that if G is a connected graph with $\nu > 2\delta$, then G has a path of length at least 2δ. (G. A. Dirac)

(Dirac, 1952 has also shown that if G is a 2-connected simple graph with $\nu \ge 2\delta$, then G has a cycle of length at least 2δ.)

4.2.10 Using the remark to exercise 4.2.9, show that every $2k$-regular simple graph on $4k+1$ vertices is hamiltonian ($k \ge 1$). (C. St. J. A. Nash-Williams)

4.2.11 G is *Hamilton-connected* if every two vertices of G are connected by a Hamilton path.

(*a*) Show that if G is Hamilton-connected and $\nu \ge 4$, then $\varepsilon \ge [\frac{1}{2}(3\nu+1)]$.

(*b*)* For $\nu \ge 4$, construct a Hamilton-connected graph G with $\varepsilon = [\frac{1}{2}(3\nu+1)]$. (J. W. Moon)

4.2.12 G is *hypohamiltonian* if G is not hamiltonian but $G - v$ is hamiltonian for every $v \in V$. Show that the Petersen graph (figure 4.4) is hypohamiltonian.

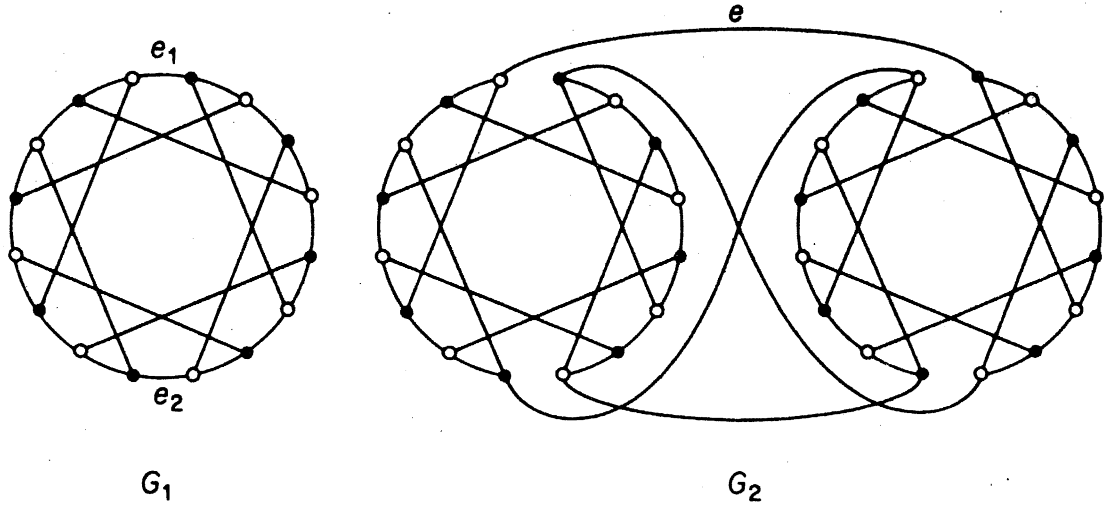
(Herz, Duby and Vigué, 1967 have shown that it is, in fact, the smallest such graph.)

4.2.13* G is *hypotraceable* if G has no Hamilton path but $G - v$ has a Hamilton path for every $v \in V$. Show that the Thomassen graph (p. 240) is hypotraceable.

4.2.14 (*a*) Show that there is no Hamilton cycle in the graph G_1 below which contains exactly one of the edges e_1 and e_2.

(*b*) Using (*a*), show that every Hamilton cycle in G_2 includes the edge e.

(*c*) Deduce that the Horton graph (p. 240) is nonhamiltonian.



4.2.15 Describe a good algorithm for
(*a*) constructing the closure of a graph;
(*b*) finding a Hamilton cycle if the closure is complete.

APPLICATIONS

4.3 THE CHINESE POSTMAN PROBLEM

In his job, a postman picks up mail at the post office, delivers it, and then returns to the post office. He must, of course, cover each street in his area at least once. Subject to this condition, he wishes to choose his route in such a way that he walks as little as possible. This problem is known as the *Chinese postman problem*, since it was first considered by a Chinese mathematician, Kuan (1962).

In a weighted graph, we define the *weight* of a tour $v_0e_1v_1 \ldots e_nv_0$ to be $\sum_{i=1}^{n} w(e_i)$. Clearly, the Chinese postman problem is just that of finding a minimum-weight tour in a weighted connected graph with non-negative weights. We shall refer to such a tour as an *optimal tour.*

If G is eulerian, then any Euler tour of G is an optimal tour because an Euler tour is a tour that traverses each edge exactly once. The Chinese postman problem is easily solved in this case, since there exists a good algorithm for determining an Euler tour in an eulerian graph. The algorithm, due to Fleury (see Lucas, 1921), constructs an Euler tour by tracing out a trail, subject to the one condition that, at any stage, a cut edge of the untraced subgraph is taken only if there is no alternative.

Fleury's Algorithm

1. Choose an arbitrary vertex v_0, and set $W_0 = v_0$.
2. Suppose that the trail $W_i = v_0e_1v_1 \ldots e_iv_i$ has been chosen.

Then choose an edge e_{i+1} from $E\backslash\{e_1, e_2, \ldots, e_i\}$ in such a way that

(i) e_{i+1} is incident with v_i;

(ii) unless there is no alternative, e_{i+1} is not a cut edge of

$$G_i = G - \{e_1, e_2, \ldots, e_i\}$$

3. Stop when step 2 can no longer be implemented.

By its definition, Fleury's algorithm constructs a trail in G.

Theorem 4.7 If G is eulerian, then any trail in G constructed by Fleury's algorithm is an Euler tour of G.

Proof Let G be eulerian, and let $W_n = v_0e_1v_1 \ldots e_nv_n$ be a trail in G constructed by Fleury's algorithm. Clearly, the terminus v_n must be of degree zero in G_n. It follows that $v_n = v_0$; in other words, W_n is a closed trail.

Suppose, now, that W_n is not an Euler tour of G, and let S be the set of vertices of positive degree in G_n. Then S is nonempty and $v_n \in \bar{S}$, where $\bar{S} = V\backslash S$. Let m be the largest integer such that $v_m \in S$ and $v_{m+1} \in \bar{S}$. Since W_n terminates in $\bar{S}$, e_{m+1} is the only edge of $[S, \bar{S}]$ in G_m, and hence is a cut edge of G_m (see figure 4.10).

Let e be any other edge of G_m incident with v_m. It follows (step 2) that e must also be a cut edge of G_m, and hence of $G_m[S]$. But since $G_m[S] = G_n[S]$, every vertex in $G_m[S]$ is of even degree. However, this implies (exercise 2.2.6*a*) that $G_m[S]$ has no cut edge, a contradiction □

The proof that Fleury's algorithm is a good algorithm is left as an exercise (exercise 4.3.2).

If G is not eulerian, then any tour in G and, in particular, an optimal tour in G, traverses some edges more than once. For example, in the graph of figure 4.11*a* *xuywvzwyxuwvxzyx* is an optimal tour (exercise 4.3.1). Notice that the four edges *ux*, *xy*, *yw* and *wv* are traversed twice by this tour.

It is convenient, at this stage, to introduce the operation of duplication of an edge. An edge e is said to be *duplicated* when its ends are joined by a

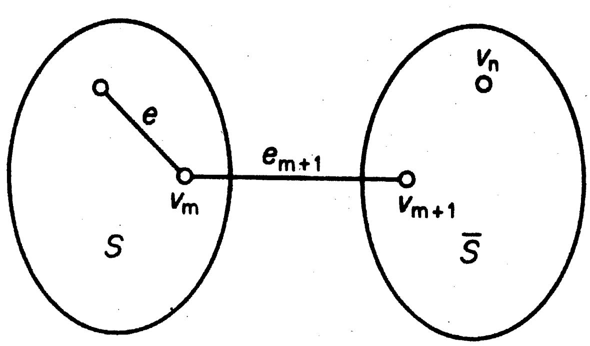


Figure 4.10

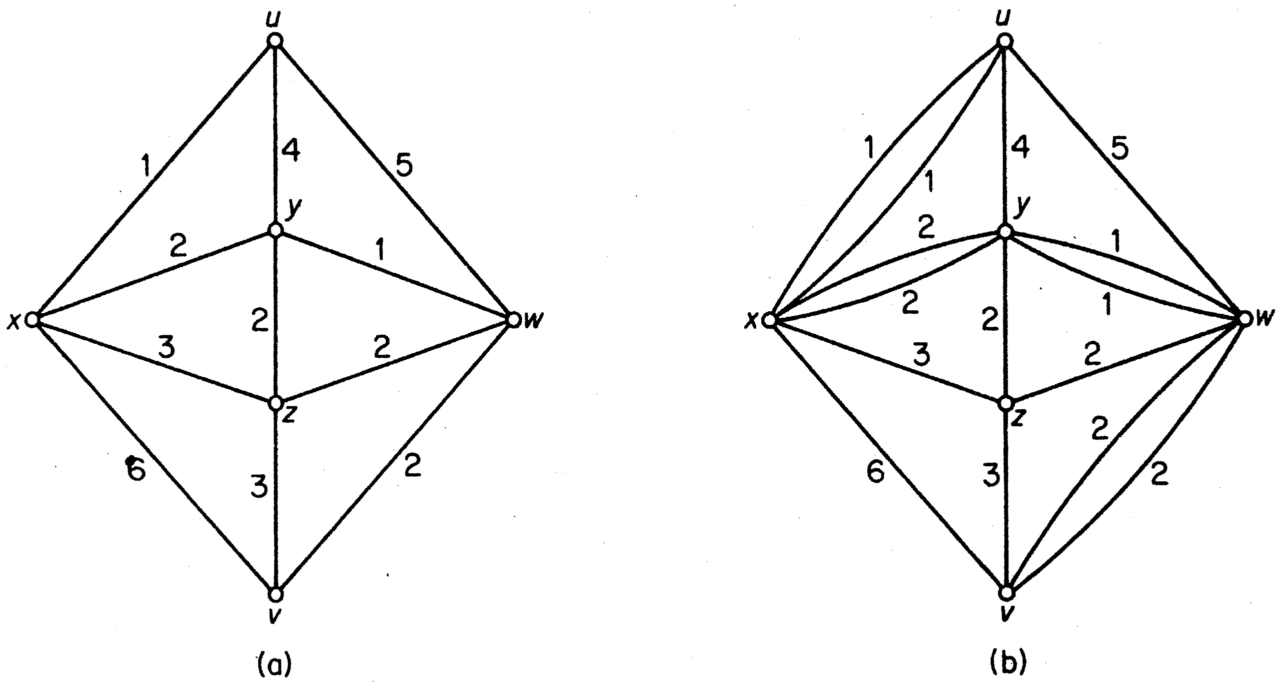


Figure 4.11

new edge of weight $w(e)$. By duplicating the edges ux, xy, yw and wv in the graph of figure 4.11*a*, we obtain the graph shown in figure 4.11*b*.

We may now rephrase the Chinese postman problem as follows: given a weighted graph G with non-negative weights,

(i) find, by duplicating edges, an eulerian weighted supergraph G^* of G such that $\sum_{e\in E(G^*)\setminus E(G)} w(e)$ is as small as possible;

(ii) find an Euler tour in G^*.

That this is equivalent to the Chinese postman problem follows from the observation that a tour of G in which edge e is traversed $m(e)$ times corresponds to an Euler tour in the graph obtained from G by duplicating e $m(e)-1$ times, and vice versa.

We have already presented a good algorithm for solving (ii), namely Fleury's algorithm. A good algorithm for solving (i) has been given by Edmonds and Johnson (1973). Unfortunately, it is too involved to be presented here. However, we shall consider one special case which affords an easy solution. This is the case where G has exactly two vertices of odd degree.

Suppose that G has exactly two vertices u and v of odd degree; let G^* be an eulerian spanning supergraph of G obtained by duplicating edges, and write E^* for $E(G^*)$. Clearly the subgraph $G^*[E^*\setminus E]$ of G^* (induced by the edges of G^* that are not in G) also has only the two vertices u and v of odd degree. It follows from corollary 1.1 that u and v are in the same component of $G^*[E^*\setminus E]$ and hence that they are connected by a (u, v)-path P^*.

Clearly

$$\sum_{e\in E^*\setminus E} w(e) \geq w(P^*) \geq w(P)$$

where P is a minimum-weight (u, v)-path in G. Thus $\sum_{e\in E^*\setminus E} w(e)$ is a minimum when G^* is obtained from G by duplicating each of the edges on a minimum-weight (u, v)-path. A good algorithm for finding such a path was given in section 1.8.

Exercises

4.3.1 Show that *xuywvzwyxuwvxzyx* is an optimal tour in the weighted graph of figure 4.11*a*.

4.3.2 Draw a flow diagram summarising Fleury's algorithm, and show that it is a good algorithm.

4.4 THE TRAVELLING SALESMAN PROBLEM

A travelling salesman wishes to visit a number of towns and then return to his starting point. Given the travelling times between towns, how should he plan his itinerary so that he visits each town exactly once and travels in all for as short a time as possible? This is known as the *travelling salesman problem*. In graphical terms, the aim is to find a minimum-weight Hamilton cycle in a weighted complete graph. We shall call such a cycle an *optimal cycle*. In contrast with the shortest path problem and the connector problem, no efficient algorithm for solving the travelling salesman problem is known. It is therefore desirable to have a method for obtaining a reasonably good (but not necessarily optimal) solution. We shall show how some of our previous theory can be employed to this end.

One possible approach is to first find a Hamilton cycle C, and then search for another of smaller weight by suitably modifying C. Perhaps the simplest such modification is as follows.

Let $C = v_1v_2 \ldots v_\nu v_1$. Then, for all i and j such that $1 < i+1 < j < \nu$, we can obtain a new Hamilton cycle

$$C_{ij} = v_1v_2 \ldots v_iv_jv_{j-1} \ldots v_{i+1}v_{j+1}v_{j+2} \ldots v_\nu v_1$$

by deleting the edges v_iv_{i+1} and v_jv_{j+1} and adding the edges v_iv_j and $v_{i+1}v_{j+1}$, as shown in figure 4.12.

If, for some i and j

$$w(v_iv_j) + w(v_{i+1}v_{j+1}) < w(v_iv_{i+1}) + w(v_jv_{j+1})$$

the cycle C_{ij} will be an improvement on C.

After performing a sequence of the above modifications, one is left with a cycle that can be improved no more by these methods. This final cycle will

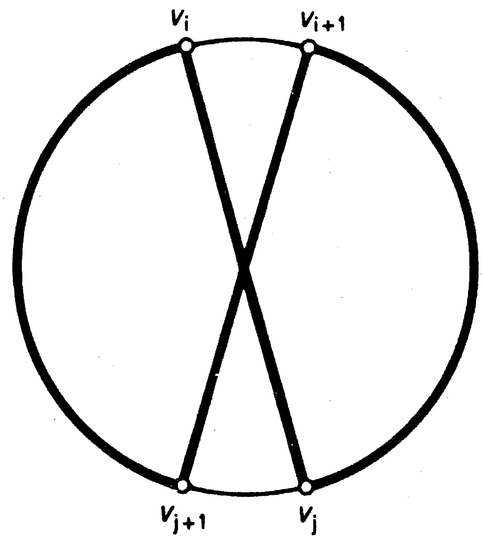


Figure 4.12

almost certainly not be optimal, but it is a reasonable assumption that it will often be fairly good; for greater accuracy, the procedure can be repeated several times, starting with a different cycle each time.

As an example, consider the weighted graph shown in figure 4.13; it is the same graph as was used in our illustration of Kruskal's algorithm in section 2.5.

Starting with the cycle L MC NY Pa Pe T L, we can apply a sequence of three modifications, as illustrated in figure 4.14, and end up with the cycle L NY MC T Pe Pa L of weight 192.

An indication of how good our solution is can sometimes be obtained by applying Kruskal's algorithm. Suppose that C is an optimal cycle in G. Then, for any vertex v, $C - v$ is a Hamilton path in $G - v$, and is therefore a

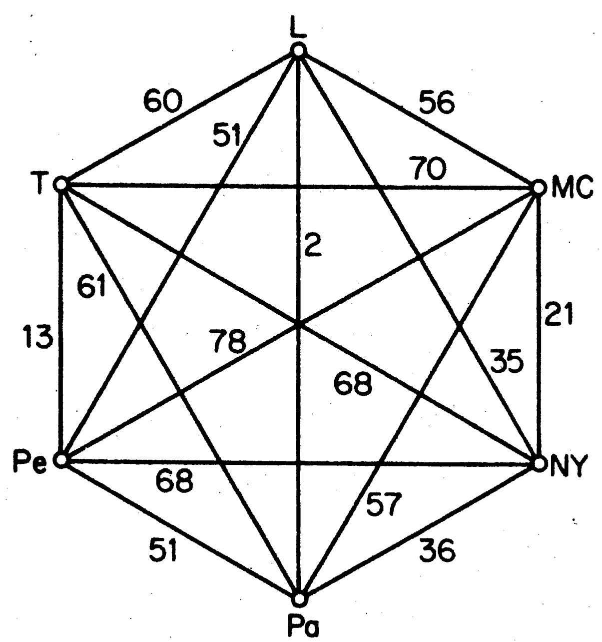


Figure 4.13

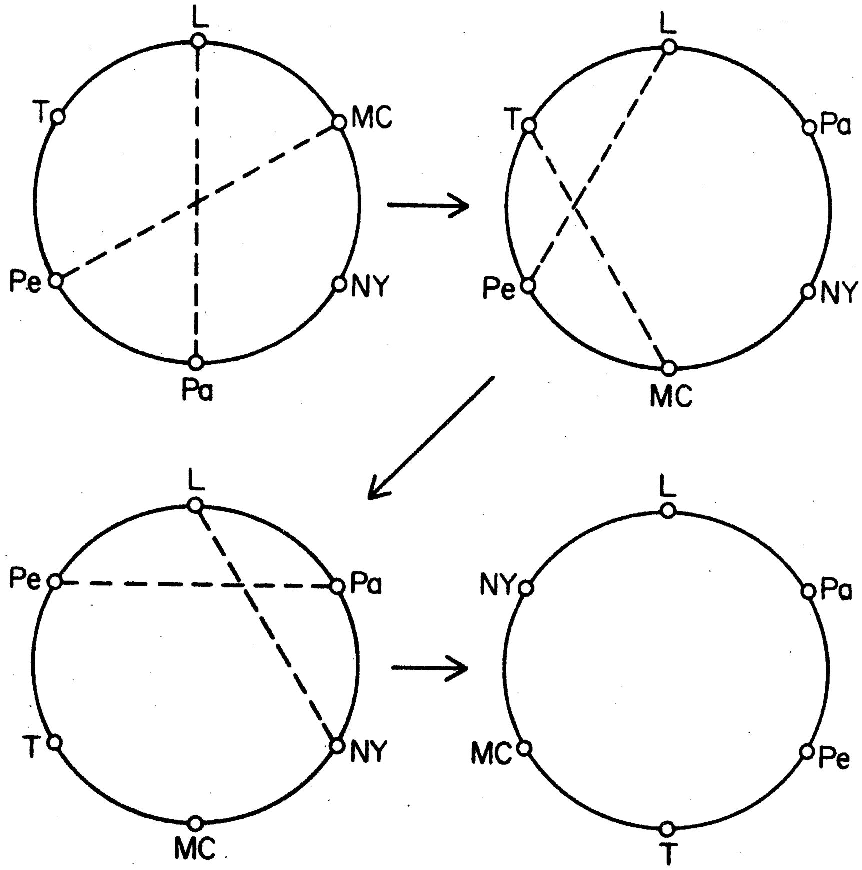


Figure 4.14

spanning tree of $G-v$. It follows that if T is an optimal tree in $G-v$, and if e and f are two edges incident with v such that $w(e)+w(f)$ is as small as possible, then $w(T)+w(e)+w(f)$ will be a lower bound on $w(C)$. In our example, taking NY as the vertex v, we find (see figure 4.15) that

$$w(T)=122 \qquad w(e)=21 \quad \text{and} \quad w(f)=35$$

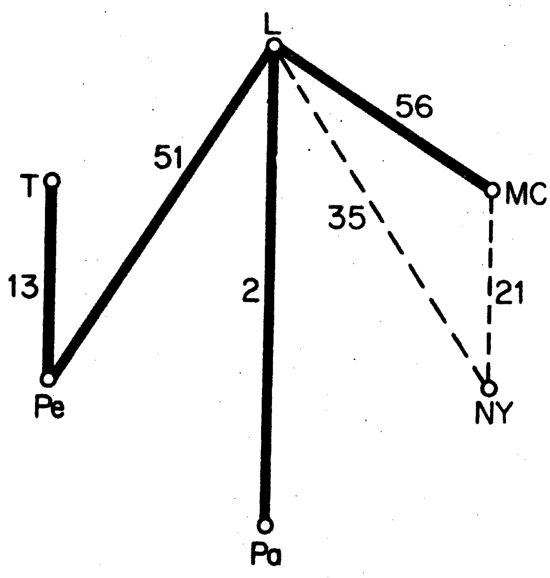


Figure 4.15

We can therefore conclude that the weight $w(C)$ of an optimal cycle in the graph of figure 4.13 satisfies

$$178 \leq w(C) \leq 192$$

The methods described here have been further developed by Lin (1965) and Held and Karp (1970; 1971). In particular, Lin has found that the cycle modification procedure can be made more efficient by replacing three edges at a time rather than just two; somewhat surprisingly, however, it is not advantageous to extend this same idea further. For a survey of the travelling salesman problem, see Bellmore and Nemhauser (1968).

Exercise

4.4.1* Let G be a weighted complete graph in which the weights satisfy the triangle inequality: $w(xy)+w(yz) \geq w(xz)$ for all x, y, $z \in V$. Show that an optimal cycle in G has weight at most $2w(T)$, where T is an optimal tree in G.

(D. J. Rosencrantz, R. E. Stearns, P. M. Lewis)

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