Wavelet Characteristics

What Wavelet Should I use?

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Abstract

This white paper briefly discusses characteristics of different wavelets in terms of their efficiencies and usefulness in different circumstances. This discussion is aimed at helping to decide what wavelet to use for a given application. The wavelets discussed in this paper include infinite support wavelets as well as finite duration wavelets. Finite support and compact wavelets are further divided into orthogonal and biorthogonal wavelets. When appropriate, the regularity of the wavelets and their symmetry are also discussed.

Introduction

Wavelet theory had been developed independently on several fronts. Different signal processing techniques, developed for signal and image processing applications, had significant contribution in this development[1]. Some of the major contributors to this theory are: multiresolution signal processing[2], used in computer vision; subband coding, developed for speech and image compression; and wavelet series expansion, developed in applied mathematics[3]. The wavelet transform is successfully applied to non-stationary signals and images. Some of the application areas are: nonlinear filtering or denoising, signal and image compression, speech coding, seismic and geological signal processing, medical and biomedical signal and image processing, and communication.

Wavelet theory is based on analyzing signals to their components by using a set of basis functions. One important characteristic of the wavelet basis functions is that they relate to each other by simple scaling and translation. The original wavelet function, known as mother wavelet, which is generally designed based on some desired characteristics associated to that function, is used to generate all basis functions. For the purpose of multiresolution formulation, there is also a need for a second function, known as scaling function, to allow analysis of the function to finite number of components. These functions and their interrelations will be discussed further in the following sections.

In most wavelet transform applications, it is required that the original signal be synthesized from the wavelet coefficients. This condition is referred to as perfect reconstruction. In some cases, however, like pattern recognition type of applications, this requirement can be relaxed. In the case of perfect reconstruction, in order to use same set of wavelets for both analysis and synthesis, and compactly represent the signal, the wavelets should also satisfy orthogonality condition. By choosing two different sets of wavelets, one for analysis and the other for synthesis, the two sets should satisfy the biorthogonality condition to achieve perfect reconstruction.

In general, the goal of most modern wavelet research is to create a mother wavelet function that will give an informative, efficient, and useful description of the signal of interest. It is not easy to design a uniform procedure for developing the best mother wavelet or wavelet transform for a given class of signals. However, based on several general characteristics of the wavelet functions, it is possible to determine which wavelet is more suitable for a given application.

In the following sections, after discussing some of the basic ideas and formulations, several popular wavelets are reviewed. Since it is not our intention to provide an extensive treatment of the subject, the interested reader is referred to the extra references in the bibliography for more in depth discussion.

Basic Concepts

A *wave* is usually referred to an oscillating function of time or space, such as sinusoid. Fourier analysis is a wave analysis, which expands signals in terms of sinusoids or equivalently complex exponentials. Wave transformation of signals has proven to be extremely valuable in mathematics, science, and engineering, especially for periodic, time-invariant, or stationary phenomenon. A *wavelet* is a small wave with finite energy, which has its energy concentrated in time or space to give a tool for the analysis of transient, nonstationary, or time-varying phenomenon. The wavelet still has the oscillating wavelike characteristics, but also has the ability to allow simultaneous time, or space, and frequency analysis with a flexible mathematical foundation. This is illustrated by an example in Figure 1.

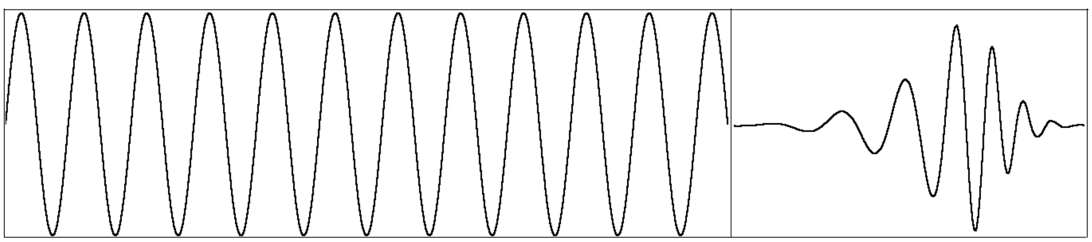

Figure 1- Comparison of a Wave and a Wavelet: Left graph is a Sine Wave with infinite energy and the right graph is a Wavelet with finite energy.

Wavelets are used to analyze signals in much the same way as complex exponentials (sine and cosine functions) used in Fourier analysis of signals. The compactness and finite energy characteristic of wavelet functions differentiate wavelet decompositions from other Fourier like analysis in their applicability to different circumstances. Wavelet functions not only can be used to analyze stationary signals but also it can be used to decompose nonstationary, time-varying or transient signals.

In this discussion, for simplicity and uniform representation, the most common dyadic and discrete formulation is discussed. For general treatment of wavelet theory, the reader is referred to many of the existing literatures on the subject some of which are referred to in the bibliography. The wavelet transform is a two-parameter expansion of a signal in terms of a particular wavelet basis functions or wavelets. Let $\psi(t)$ represent the mother wavelet. All other wavelets are obtained by simple *scaling* and *translation* of $\psi(t)$ as follows:

$$\psi_{a,\tau}(t) = \left(1/\sqrt{a}\right)\psi\left[\left(t-\tau\right)/a\right] \tag{1}$$

In the most common formulation, the scaling is discrete and dyadic, $a = 2^{-j}$. The translation is discretized with respect to each scale by using $\tau = k2^{-j}T$. In this case, the wavelet basis functions are obtained by

$$\psi_{j,k}(t) = 2^{j/2}\psi\left(2^{j}t - kT\right) \tag{2}$$

for different integer values of j and k. Integer k represents translation of the wavelet function and is an indication of time or space in wavelet transform. Integer j, however, is an indication of the wavelet frequency or spectrum shift and generally referred to as *scale*. This parameterization of the time (or space) by integer k and frequency or scale by integer j turns out to be extremely effective. For demonstration purposes, two different scaled versions of a wavelet along with the mother wavelet are shown in Figure 2.

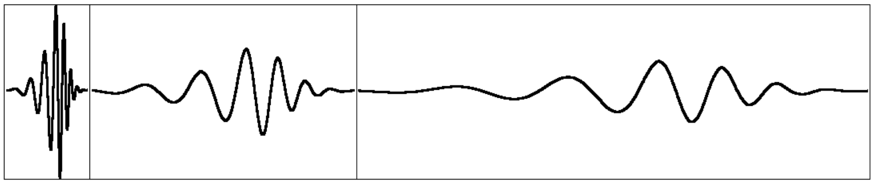

Figure 2- The left graph is the mother wavelet ψ_{D12}, the middle one is the wavelet at scale $j=-1$ and the right one is the wavelet at scale $j=-2$. The other way to look at these graphs is: to assume that the right graph is the mother wavelet, the middle one is the wavelet at scale $j=1$ and the left one is the wavelet at scale $j=2$.

As it is evident from Figure 2, different scales referred to different frequency spectrum. If we only look at the center frequency of each spectrum, it is easy to see that the center frequency changes by a factor of two for each increment or decrement of integer scale j. For example, if the center frequency of the mother wavelet is at f_o, then the center frequency of the wavelet with integer scale j is $\left(2^j \cdot f_o\right)$. For simplicity, in wavelet transform reference to frequency is replaced by reference to scale. Another aspect of the wavelet transform is that the localization or compactness of the wavelet increases as frequency or scale increases. In other words, higher scale corresponds to finer localization and vice versa.

The multiresolution formulation needs two closely related basic functions. In addition to the wavelet $\psi(t)$, there is a need for another basic function called the *scaling function*, which is denoted by $\varphi(t)$. Scaling and translation of $\varphi(t)$ is defined similar to (1) and (2). Without getting into the theoretical discussions and with reference to [4], the two-parameter wavelet expansion for signal $x(t)$ is given by the following decomposition series in which the scaling and wavelet functions are utilized.

$$x(t) = \sum_{k} c_k \varphi_{j_o,k}(t) + \sum_{k} \sum_{j=j_o}^{\infty} d_{j,k} \psi_{j,k}(t) \quad (3)$$

In this expansion, c_k coefficients are referred to as *approximation* coefficients at scale j_o. The set of $d_{j,k}$ coefficients represents *details* of the signal at different scales. The discrete wavelet transform (DWT) coefficients consist of both c_k's and $d_{j,k}$'s. In this case, since the signal is continuous, the upper limit for the scales of the details can go to infinity. However, for discrete signals, this upper limit is bounded to the maximum available details in the discrete signal.

Relations of the wavelet coefficients to the original signal, for real and orthogonal wavelets, are given by the following two equations.

$$d_{j,k} = \int x(t) \psi_{j,k}(t) dt \quad (4)$$

$$c_k = \int x(t) \varphi_{j_o,k}(t) dt \quad (5)$$

If the wavelets are biorthogonal, wavelet and scaling functions appear in pairs, $\psi(t)$, $\tilde{\psi}(t)$ and $\varphi(t)$, $\tilde{\varphi}(t)$. In this case, one set of the wavelet and scaling functions is used for analysis and the other set is used for synthesis. In other words, if (3) is given for synthesis, then the wavelet coefficients are obtained from the following two equations.

$$d_{j,k} = \quad x(t)\tilde{\psi}_{j,k}(t)dt \tag{6}$$

$$c_k = \quad x(t)\tilde{\varphi}_{j_o,k}(t)dt \tag{7}$$

Efficient calculation of the DWT coefficients is generally formulated in terms of a particular set of multirate filters. Filters used for calculation of the forward transform are referred to as analysis filters and those used for calculation of the inverse transform are referred to as synthesis filters. The coefficients of these filters, which are generally FIR, are obtained from the knowledge of the mother wavelet and scaling functions. Computational formulas and other related topics are discussed in [4].

Unlike Fourier-like transforms, which are generally based on a particular set of basis functions, there exist many different wavelet bases with different characteristics. Success of a given wavelet basis in a particular application does not necessarily mean that this set is efficient at other applications. Therefore, the freedom for choosing a particular wavelet for an application should carefully be explored.

Popular Wavelets

The simplest wavelet also referred to as Haar wavelet, turns out to be the only orthogonal wavelet that has symmetric analysis and synthesis filters. This particular wavelet has been studied extensively in the image processing area as Haar transform. Graphs of Haar scaling function and mother wavelet are shown in Figure 3. This particular wavelet is ideal in situations with limited computational resources.

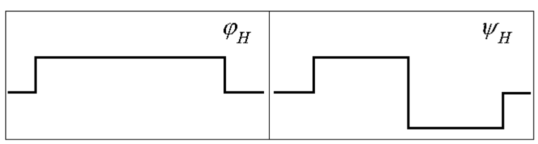

Figure 3- Left graph is the Haar scaling function and the right one is the Haar mother wavelet.

Due to simplicity and existence of fast computational algorithm, historically, Haar transform was a good choice for image processing. Advances in high speed VLSI not only provide the opportunity to utilize the Fourier transform in real-time processing of signals and images, but also provide opportunities to implement and explore new and more advanced signal and image processing algorithms.

Researchers in Applied Mathematics, Communications, and Signal/Image Processing areas have developed many different wavelet systems and some are still actively working in designing even newer wavelets with specialized characteristics. Wavelets can be divided in different classes in many different ways. For example, we can divide them based on their duration or support: infinite support wavelets and finite duration wavelets. There are several interesting wavelets with infinite support. Some of the infinite support wavelets are Gaussian wavelets, Mexican Hat, Morlet, and Meyer. Gaussian wavelets are obtained from derivatives of the Gaussian function. Several examples of the Gaussian wavelets along with other infinite duration wavelets are shown in Figure 4. Mexican Hat wavelet, referred to as ψ_{MH} , is similar to the Gaussian wavelet ψ_{G2} . Among these wavelets, only Meyer wavelet has a scaling function.

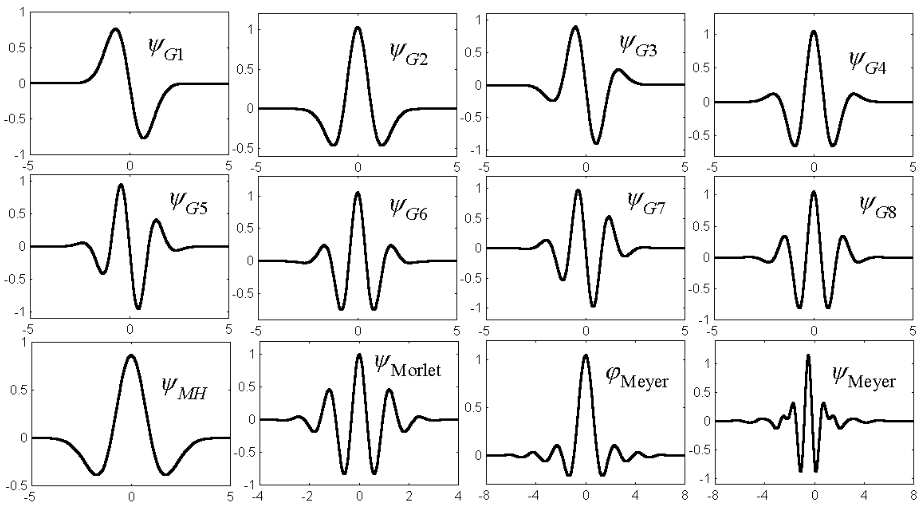

Figure 4- Different Gaussian wavelets obtained from derivatives of the Gaussian function along with Mexican Hat wavelet, Morlet wavelet and Meyer scaling function and wavelet. The order of the derivatives for Gaussian wavelets are shown as subscript for these wavelets.

In practice, finite support and compact wavelets are more popular due to their relations to multiresolution filter banks. These wavelets have finite impulse response (FIR) wavelet filters. Among these wavelets, the most commonly used can be categorized into two classes: orthogonal and biorthogonal wavelet systems. Orthogonal wavelets decompose signals into well- behaved orthogonal signal spaces. In this case, however, the analysis and synthesis filters are not symmetric, a condition that might be required in some applications like image processing. Biorthogonal wavelets are more complicated and are defined based on a pair of scaling and wavelet functions. Due to more flexibility in this case, the analysis and synthesis filters can be forced to be symmetric and hence be useful for applications that demand linear phase filtering.

Orthogonality property is the most desired property in any signal analysis operation. For orthogonal wavelet systems with real functions, the following conditions should be satisfied.

$$\psi_{j,k}(t)\cdot\psi_{m,n}(t)dt = \begin{matrix} 1 & \text{if } j=m \text{ and } k=n \\ 0 & \text{Otherwise} \end{matrix} \tag{8}$$

$$\varphi_{j,k}(t)\cdot\varphi_{m,n}(t)dt = \begin{matrix} 1 & \text{if } j=m \text{ and } k=n \\ 0 & \text{Otherwise} \end{matrix} \tag{9}$$

$$\varphi_{j,k}(t)\cdot\psi_{j,}\ (t)dt = 0 \tag{10}$$

Many functions exist that can satisfy the orthogonality requirements. Some of these functions are extraordinarily irregular, even fractal in nature. This may be an advantage in analyzing rough or fractal signals but it is likely to be a disadvantage for most signals and images. It has been shown that the number of vanishing moments of the wavelet, $\psi(t)$, is related to the smoothness or differentiability of $\varphi(t)$ and $\psi(t)$. The representation and approximation of polynomials, which are often a good model for certain signals and images, are also related to the number of vanishing or minimized wavelet moments. On the other hand, the number of zero moments in the scaling function, $\varphi(t)$, is related to the goodness of the approximation of high resolution scaling coefficients by samples of the signal. This number also affects the symmetry and concentration of the scaling functions and wavelets.

For a given wavelet order, order of the wavelet FIR filter, Daubechies developed wavelets with maximum regularity. In this case, the number of zero moments for $\psi(t)$ is maximized. These wavelets, which are referred to as Daubechies wavelets, are fully parameterized and have straightforward procedure for calculation of their analysis filters. Several examples of Daubechies' wavelets and scaling functions are shown in Figure 5. The order of the wavelet filter for orthogonal wavelets is always an even number. In Figure 5, the order of each wavelet filter is shown by the indices used on the corresponding $\varphi(t)$ and $\psi(t)$ functions.

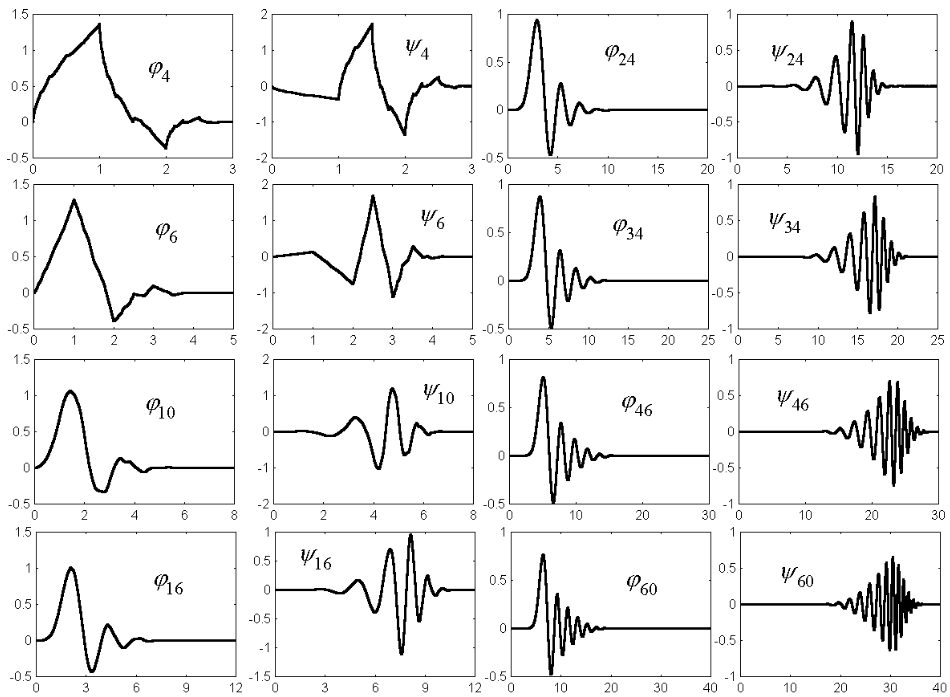

Figure 5- Several examples of Daubechies scaling and wavelet functions.

Daubechies wavelets have good compression property for wavelet coefficients but not for approximation coefficients. In other words, with reference to (3), the wavelet coefficients $d_{j,k}$'s have the best compression property under the circumstances while c_k's do not. In order to resolve this issue, Coifman suggested to have as many zero moments for scaling function as for wavelets. This modification resulted in what is referred to as coiflets. Figure 6 shows several of the examples in this case. These wavelets have similar compression characteristics in both their approximation and detail coefficients. In Figure 6, the order of each wavelet filter is shown by the indices used on the corresponding $\varphi(t)$ and $\psi(t)$ functions.

In general, all orthogonal wavelets are asymmetric. For some applications, it does not really matter if the wavelet is symmetric or not. However, in other applications this may be a nuisance. In image processing applications, e.g., image coding, since human vision is more tolerant to symmetric error than asymmetric one, it is very desirable to use symmetric wavelets. In addition, symmetric wavelets make it easier to deal with the boundaries of the image. Daubechies has shown that with some modifications, it is possible to design orthogonal wavelets that are least symmetric. Interested readers are referred to [3] for further discussion on this issue. By comparison, coiflets are closer to symmetry but still are not perfectly symmetric. Perfect symmetry is possible only for complex wavelet filters, biorthogonal wavelets, infinite support wavelets, and multi-wavelets.

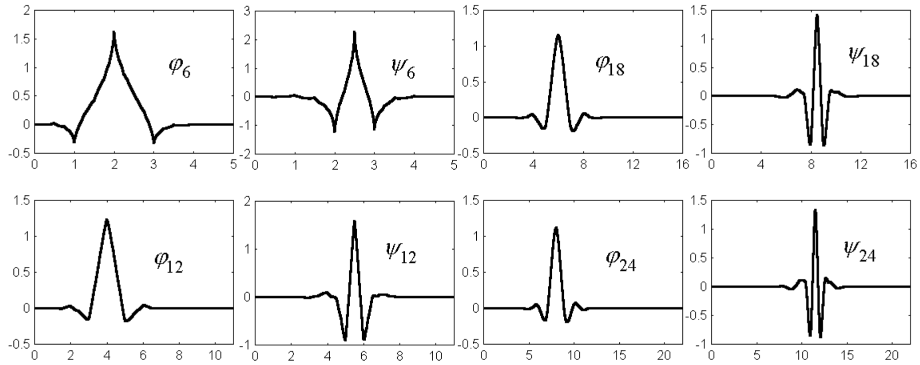

Figure 6- Several examples of Coiflets scaling and wavelet functions.

For most applications, it is desired to have real filter coefficients. In these cases, the only choice for the class of symmetric wavelets would be biorthogonal wavelets. The most common biorthogonal wavelets are those based on spline functions. Several examples of these wavelets are shown in Figure 7.

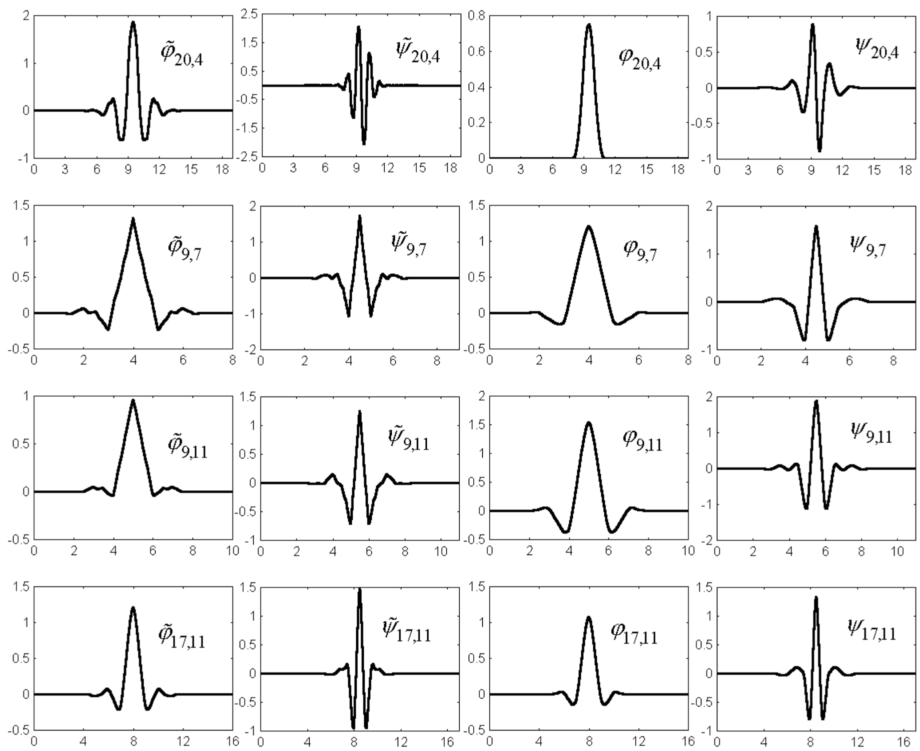

Figure 7- Several examples of biorthogonal wavelets.

In this case, for each wavelet system, there are two scaling functions, $\varphi(t)$ and $\tilde{\varphi}(t)$, and two wavelet functions, $\psi(t)$ and $\tilde{\psi}(t)$. Therefore in each case, there are two wavelet filters associated to $\psi(t)$ and $\tilde{\psi}(t)$. This association is shown by two indices on each function. The first number represents the order of the lowpass filter for analysis and the second number represents the order of the lowpass filter for synthesis. Unlike the orthogonal case, for biorthogonal wavelets, the order of the filters can be odd or even and they do not have to be the same. The only restriction in this case is that their difference should be an even number. In other words, both filters should be odd or they both should be even. It should also be pointed out that the roles of analysis and synthesis filters are interchangeable.

Applications

Wavelet transform has found many applications in applied mathematics and signal processing. Due to its zooming property, which allows a very good representation of discontinuities, wavelet transform is successfully used in solving partial differential equations. They give a generalization of finite element method and, due to their localizing ability, provide sparse operators and good numerical stability.

In seismic and geological signal processing as well as medical and biomedical signal and image processing, wavelet transforms are used for denoising, compression, and detection. In general, there are many examples of successful application of wavelets in signal and image processing: speech coding, communications, radar, sonar, denoising, edge detection, and feature detection. Wavelet transform is also used in multi-scale models of stochastic processes and analysis and synthesis of $1/f$ noise.

Orthogonal wavelets are very successful in numerical analysis like solving partial differential equations, speech coding and other similar applications, where symmetry is not a major requirement. Daubechies wavelets are very good in terms of their compact representation of signal details. They are, however, not efficient in representation of signal approximation at a given resolution. On the other hand, coiflets are similarly effective for both signal details and signal approximation. In image processing applications, biorthogonal wavelets, which are symmetric, are more desirable. Symmetric wavelets allow extension at the image boundaries and prevent image contents from shifting between subbands. In this case, due to human sensitivity to asymmetric errors, orthogonal wavelets usually are not used. For FBI digitized fingerprints compression[5], it is found that the biorthogonal wavelet system represented by $\varphi_{9,7}(t)$, $\tilde{\varphi}_{9,7}(t)$, $\psi_{9,7}(t)$, and $\tilde{\psi}_{9,7}(t)$, in Figure 7, is very successful.

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