

## Sequence

- A sequence is a list of objects arranged in a definite order
- Difference between set and sequence
- A set has no order and no duplicated elements
- A sequence has a specific order and elements may be duplicated
- Nomenclature: $\mathrm{a}_{1}, \mathrm{a}_{2}, \ldots \mathrm{a}_{\mathrm{n}}$


## Types of Sequences

- Sequence may stop after n steps (finite) or go on for ever (infinite)
- Formulas can be used to describe sequences
- Recursive
- Explicit
- The sequences $1,2,3,2,2,3,1$ and 2,1 , $3,2,2,3,1$ only switch two elements, but that alone makes them unequal


## Recursive Sequences

In a recursive sequence, the next item in the sequence is determined from previous values
Difficult to determine say $100^{\text {th }}$ element since previous 99 need to be determined first.
Example:
$\mathrm{a}_{1}=1$
$a_{2}=2$
$a_{n}=a_{n-1}+a_{n-2}$

## Explicit Sequences

- In an explicit sequence, the nth item is determined by a formula depending only on $n$
- Easier to determine any element
- Example: $\mathrm{a}_{\mathrm{n}}=2 \times n$

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## Strings

- Sequences can be made up of characters too
- Example: W, a, k, e, , u, p
- Removing the commas and you get a string: "Wake up"
- Strings best illustrate difference between sequences and sets
$-a, b, a, b, a, b, a, \ldots$ is a sequence, i.e., "abababa..." is a string
- The corresponding set is $\{a, b\}$


## Characteristic Functions

- A characteristic function is a function defining membership in a set
- $f_{A}(x)= \begin{cases}1 & \text { if } x \in A \\ 0 & \text { if } x \notin A\end{cases}$
- Example, for the set $A=\{1,4,6\}$
$f_{A}(1)=1, f_{A}(2)=0, f_{A}(3)=0, f_{A}(4)=1$, etc.

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## Properties of Characteristic Functions

Characteristic functions of subsets satisfy the following properties (proofs are on page 15 of textbook.)
$-f_{A \cap B}=f_{A} f_{B}$; that is $f_{A \cap B}(x)=f_{A}(x) f_{B}(x)$ for all $x$.
$-f_{\mathrm{A} \cup \mathrm{B}}=f_{\mathrm{A}}+f_{\mathrm{B}}-f_{\mathrm{A}} f_{\mathrm{B}}$;
that is $f_{A \cup B}(x)=f_{A}(x)+f_{B}(x)-f_{A}(x) f_{B}(x)$ for all $x$.
$-f_{\mathrm{A} \oplus \mathrm{B}}=f_{\mathrm{A}}+f_{\mathrm{B}}-2 f_{\mathrm{A}} f_{\mathrm{B}}$;
that is $f_{A \oplus B}(x)=f_{A}(x)+f_{B}(x)-2 f_{A}(x) f_{B}(x)$ for all $x$.

## Programming Example

Characteristic functions may look unfamiliar, but consider the following code:
if (insert conditional statement here) return (1);
else return (0);

Example: $A=\{x \mid x>4\}$
if $(x>4)$ return (1); else return (0);

## Proving Characteristic Function Properties

- Alternate way of doing proof is to enumerate all four cases and see how the result comes out
- Example: Prove $f_{\mathrm{A} \cap \mathrm{B}}=f_{\mathrm{A}} f_{\mathrm{B}}$


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\begin{aligned}
& f_{\mathrm{A}}(\mathrm{a})=0, f_{\mathrm{B}}(\mathrm{a})=0, f_{\mathrm{A}}(\mathrm{a}) \times f_{\mathrm{B}}(\mathrm{a})=0 \times 0=0=\mathrm{f}_{\mathrm{A} \cap \mathrm{~B}}(\mathrm{a}) \\
& f_{\mathrm{A}}(\mathrm{~b})=0, f_{\mathrm{B}}(\mathrm{~b})=1, f_{\mathrm{A}}(\mathrm{~b}) \times f_{\mathrm{B}}(\mathrm{~b})=0 \times 1=0=0=\mathrm{f}_{\mathrm{A} \cap \mathrm{~B}}(\mathrm{~b}) \\
& f_{\mathrm{A}}(\mathrm{c})=1, f_{\mathrm{B}}(\mathrm{c})=0, f_{\mathrm{A}}(\mathrm{c}) \times f_{\mathrm{B}}(\mathrm{c})=1 \times 0=0=\mathrm{f}_{\mathrm{A} \cap \mathrm{~B}}(\mathrm{c}) \\
& f_{\mathrm{A}}(\mathrm{~d})=1, f_{\mathrm{B}}(\mathrm{~d})=1, f_{\mathrm{A}}(\mathrm{~d}) \times f_{\mathrm{B}}(\mathrm{~d})=1 \times 1=1=1=\mathrm{f}_{\mathrm{A} \cap \mathrm{~B}}(\mathrm{~d})
\end{aligned}
$$

## Representing Sets with a Computer

- Remember that sets have no order and no duplicated elements.
- The general need to assign each element in a set to a memory location gives it order in a computer.
- We can use the characteristic function to define a set using a computer.

Representing Sets with a Computer (continued)

- Assume $U$ defines a finite universal set $U=\left\{x_{1}\right.$, $\left.x_{2}, x_{3}, \ldots, x_{n}\right\}$
- We can use characteristic function to represent subsets of $U$
- $f_{A}(x)$ is a sequence of 1 's and 0 's with the same number of elements as $U$
- $f_{\mathrm{A}}(\mathrm{x})=1$ is in position if corresponding element of $U, \mathrm{x}$, is a member of $A$
- $f_{\mathrm{A}}(\mathrm{x})=0$ is in position if corresponding element of $U, x$, is not a member of $A$


## Representing Sets with a Computer Example

- $U=\{0,1,2,3,4,5,6,7,8,9\}$
- $f_{A}(x)=1,0,1,0,0,1,1,0,0,1$
- $A=\{0,2,5,6,9\}$


## More Properties of Sequences

- Any set with n elements can be arranged as a sequence of length $n$, but not vice versa. This is because sets have no order and duplicates are not allowed.
- Each subset can be identified with its characteristic function as a sequence of 1's and 0's.
- Characteristic function for universal set, U , is a sequence of all ones.


## Strings and Regular Expressions

- Given a set $A$, the set $A^{*}$ consists of all finite sequences of elements of $A$
- Example:
$-A=$ alphabet $=\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s$, $\mathrm{t}, \mathrm{u}, \mathrm{v}, \mathrm{w}, \mathrm{x}, \mathrm{y}, \mathrm{z}\}$
$-A^{*}=$ words (the finite sequences, words, in A are not written with commas)
- $A^{*}$ contains all possible words, even those that are unpronounceable or make no sense such as "prsartkc"
- The empty sequence or empty string is represented with $\Lambda$
- Best example is real numbers
- E.g., what comes after 1.23534 ?

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## Countable and Uncountable

- A set is countable if it corresponds to some sequence.
- Members of set can be arranged in a list
- Elements have position
- All finite sets are countable
- Not all infinite sets are countable, and are therefore uncountable


## Catenation

- Two strings may be joined into a single string
- Assume $w_{1}=s_{1} s_{2} s_{3} s_{4} \ldots s_{\mathrm{n}}$ and $w_{2}=t_{1} t_{2} t_{3} t_{4} \ldots t_{\mathrm{k}}$
- The catenation of $w_{1}$ and $w_{2}$ is the sequence $s_{1} s_{2} s_{3} s_{4} \ldots s_{\mathrm{n}} t_{1} t_{2} t_{3} t_{4} \ldots t_{\mathrm{k}}$
- Notation: catenation of $w_{1}$ and $w_{2}$ is written as $w_{1} \cdot w_{2}$ or $w_{1} w_{2}$


## Some Properties of Catenation

- If $w_{1} \cdot w_{2}$ are elements of $A^{\star}$, then $w_{1} \cdot w_{2}$ is an element of $\mathrm{A}^{*}$
- $w \cdot \Lambda=w$ and $\Lambda \cdot w=w$
- A subset $B$ of $A^{*}$ has its own set $B^{*}$ which contains sentences made up from the words of $A$.
- For example:
$B=\{$ John, Jane, swims, runs, well, quickly, slowly $\}$ is a subset of $A^{*}$ where $A=$ alphabet
The element "Jane swims quickly" is an element of $B^{\star}$.


## Regular Expressions (continued)

- A regular expression on a set $A$ is a recursive formula for a sequence.
- It is made up of the elements of $A$ and the symbols (, ), $\vee$, *, $\Lambda$
- The symbol $\Lambda$ is a regular expression
- If $x \in A$, the symbol $x$ is a regular expression.
- If $\alpha$ and $\beta$ are regular expressions, then the expression $\alpha \beta$ is regular.
- If $\alpha$ and $\beta$ are regular expressions, then the expression $(\alpha \vee \beta)$ is regular.
- If $\alpha$ is a regular expression, then the expression $(\alpha)^{*}$ is regular.


## Rules of Regular Expressions

- The expression $\Lambda$ corresponds to the set $\{\Lambda\}$, where $\Lambda$ is the empty string in $A^{*}$.
- If $x \in A$, then the regular expression $x$ corresponds to the set $\{x\}$
- If $\alpha$ and $\beta$ are regular expressions corresponding to the subsets $M$ and $N$ of $A^{*}$, then $\alpha \beta$ corresponds to $M \cdot N=\{s \cdot t \mid s \in M$ and $t \in N\}$. Therefore, $M \cdot N$ is the set of all catenations of strings in $M$ with strings in $N$.


## Rules of Regular Expressions (continued)

- If the regular expressions $\alpha$ and $\beta$ correspond to the subsets $M$ and $N$ of $A^{*}$, then ( $\alpha \vee \beta$ ) corresponds to $M \cup N$.
- If the regular expression $\alpha$ corresponds to the subset $M$ of $A^{*}$, then ( $\left.\alpha\right)^{*}$ corresponds to the set $M^{*}$. Note that $M$ is a set of strings from $A$. Elements from $M^{*}$ are finite sequences of such strings, and thus may themselves be interpreted as strings from $A$. Note also that we always have $\Lambda \in M^{*}$.

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