Points missed: _____ Student's Name: ____

Total score: _____/100 points

East Tennessee State University – Department of Computer and Information Sciences
CSCI 2710 (Tarnoff) – Discrete Structures
TEST 1 for Spring Semester, 2005

Read this before starting!

- This test is closed book and closed notes
- You may ***NOT*** use a calculator
- All answers must have a box drawn around them. This is to aid the grader (who might not be me!) Failure to do so might result in no credit for answer.
- If you perform work on the back of a page in this test, indicate that you have done so in case the need arises for partial credit to be determined.
- ***If not otherwise identified, every integer is assumed to be represented using base 10.***
- Statement regarding academic misconduct from Section 5.7 of the East Tennessee State University Faculty Handbook, June 1, 2001:

  "Academic misconduct will be subject to disciplinary action. Any act of dishonesty in academic work constitutes academic misconduct. This includes plagiarism, the changing of falsifying of any academic documents or materials, cheating, and the giving or receiving of unauthorized aid in tests, examinations, or other assigned school work. Penalties for academic misconduct will vary with the seriousness of the offense and may include, but are not limited to: a grade of 'F' on the work in question, a grade of 'F' of the course, reprimand, probation, suspension, and expulsion. For a second academic offense the penalty is permanent expulsion."

Algebraic properties of sets:

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cup A = A$
- $A \cap A = A$
- $\overline{(\overline{A})} = A$
- $\overline{A} \cup A = U$
- $\overline{A} \cap A = \varnothing$
- $\overline{\varnothing} = U$
- $\overline{U} = \varnothing$
- $\overline{A \cup B} = \overline{A} \cap \overline{B}$
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- $A \cup U = U$
- $A \cap U = A$
- $A \cup \varnothing = A$ or $A \cup \{\ \} = A$
- $A \cap \varnothing = \varnothing$ or $A \cap \{\ \} = \{\ \}$

Addition principle of cardinality:

- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Properties of characteristic functions:

- $f_{A\cap B} = f_A f_B$; that is $f_{A\cap B}(x) = f_A(x) f_B(x)$ for all x.
- $f_{A\cup B} = f_A + f_B - f_A f_B$; that is $f_{A\cup B}(x) = f_A(x) + f_B(x) - f_A(x) f_B(x)$ for all x.
- $f_{A\oplus B} = f_A + f_B - 2f_A f_B$; that is $f_{A\oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x) f_B(x)$ for all x.

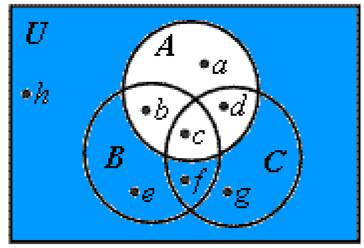
Properties of integers:

- If n and m are integers and $n > 0$, we can write $m = qn + r$ for integers q and r with $0 \leq r < n$.
- Properties of divisibility:
  - If a | b and a | c, then a | (b + c)
  - If a | b or a | c, then a | bc
  - If a | b and a | c, where b > c, then a | (b - c)
  - If a | b and b | c, then a | c
- Every positive integer $n > 1$ can be written uniquely as $n = p_1^{k1} p_2^{k2} p_3^{k3} p_4^{k4} \ldots p_s^{ks}$ where $p_1 < p_2 < p_3 < p_4 < \ldots < p_s$ are distinct primes that divide n and the k's are positive integers giving the number of times each prime occurs as a factor of n.

Problems 1 through 4 refer to the Venn Diagram shown to the right. (2 points each)

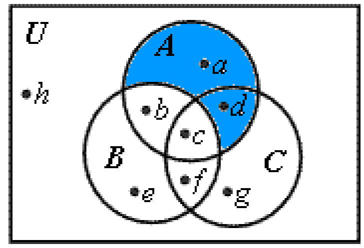
1. True or False: $f \notin \overline{\boldsymbol{A}}$

The shaded area of the Venn Diagram to the right represents $\overline{\boldsymbol{A}}$, and since *f* is contained in this area, the answer is FALSE.



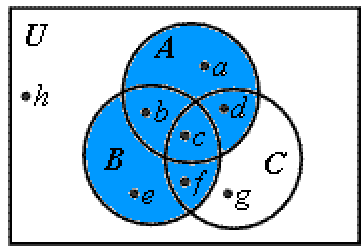
2. True or False: $d \in \boldsymbol{A} - \boldsymbol{B}$

The shaded area of the Venn Diagram to the right represents ***A-B***, and since *d* is contained in this area, the answer is TRUE.



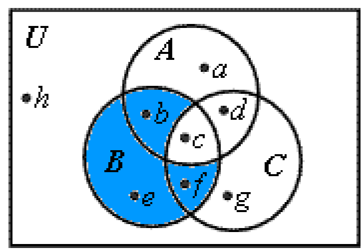
3. True or False: $d \in \boldsymbol{A} \cup \boldsymbol{B}$

The shaded area of the Venn Diagram to the right represents $\boldsymbol{A} \cup \boldsymbol{B}$, and since *d* is contained in this area, the answer is TRUE.



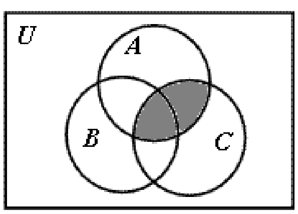
4. True or False: $b \in \boldsymbol{B} - (\boldsymbol{A} \cap \boldsymbol{B} \cap \boldsymbol{C})$

The shaded area of the Venn Diagram to the right represents $\boldsymbol{B} - (\boldsymbol{A} \cap \boldsymbol{B} \cap \boldsymbol{C})$, and since *b* is contained in this area, the answer is TRUE.



For problems 5, 6, and 7, use unions, intersections, and complements of the sets ***A***, ***B***, and ***C***, to write an expression to describe the set represented by the shaded area in the given Venn Diagram. (3 points each)

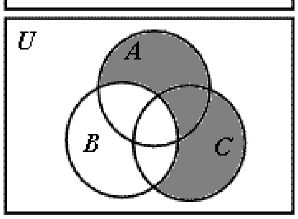
5.



The shaded area is the region where A and C overlap, and therefore represents the intersection of the two circles.

Answer for 5: $\boldsymbol{A} \cap \boldsymbol{C}$

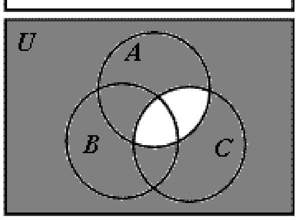
6.



The shaded area is the region included by either A or C (the union) minus the area of B.

Answer for 6: $(\boldsymbol{A} \cup \boldsymbol{C}) - \boldsymbol{B}$

7.



The shaded area is the region of everything ***but*** where A and C overlap, and therefore represents the complement of intersection of the two circles, i.e., the complement of problem 5.

Answer for 7: $\overline{\boldsymbol{A} \cap \boldsymbol{C}}$ or $\boldsymbol{U} - (\boldsymbol{A} \cap \boldsymbol{C})$

8. If ***A*** = {1, 2, 3, 4, 5} and ***B*** = {4, 5, 6, 7, 8}, compute the set represented by ***A∩B***. (3 points)

***A∩B*= {4, 5}** – This is the set of all values that are members of both *A* and *B*.

9. If ***A*** = {1, 2, 3, 4, 5} and ***B*** = {4, 5, 6, 7, 8}, compute the set represented by ***A⊕B***. (3 points)

***A⊕B*= {1, 2, 3, 6, 7, 8}** – This is the set of all values that are members of either *A* or *B*, but not both.

10. If ***A*** = {1, 2, 3, 4, 5} and ***U*** = {0, 1, 2, 3, 4, 5, 6, 7, 8}, compute the set represented by $\bar{A}$. (3 points)

$\bar{A}$ **= {0, 6, 7, 8}** – This is the set of all values in the universal set U that are not members of *A*.

11. If set ***A*** has 32 elements, set ***B*** has 40 elements, and they have 5 elements in common, how many elements are a member of ***A∪B***? (3 points)

The number of elements of ***A*** ∪ ***B*** is equivalent to the number of elements of ***A*** that are not in ***B*** plus the number of elements of ***B*** that are not in ***A*** plus the number of elements that are in both ***A*** and ***B***. If you simply add 32 to 40, you've counted the last group, the number of elements that are in both ***A*** and ***B***, twice. Therefore, we need to subtract 5 in order to arrive at the answer of 67. Many students confused this with the problem from last year's test that asked for the number of elements in ***A*** and ***B*** **but not both *A*** and ***B***. That would've meant we needed to subtract the 5 twice. This would have yielded the wrong answer for this problem.

Answer: _____67_____

12. Give three different sequences that have {a, d, s} as the corresponding set. (3 points)

The number of possible answers to this question is infinite. Any sequence using the letters a, d, and s is valid including the empty sequence. Some examples would be:

asd, ads, dsa, das, sad, sda, d, a, s, ss, dd, aa, sss, ddd, aaa, asa, da, and so on.

13. Write a formula for the n-th term of the sequence 1, 6, 11, 16, 21, 26, … (3 points)

Each element is five greater than the previous element with the first element equaling 1. This can be described as either an explicit sequence or a recursive sequence. As an explicit sequence, the n-th term is described as:

$$A_n = 5n - 4$$

As a recursive sequence, the n-th element is the n-1 element plus 5. If you are going to use the recursive form, you ***must*** show what the first element is in order to give a starting point. The formula alone is not enough.

$$A_n = A_{n-1} + 5;\ A_1 = 1;\ n>1$$

For problems 14, 15, and 16, determine if the sequence defined by the formula is explicit or recursive. In addition, identify the value of the 3rd element in the sequence. (3 points each)

14. $a_n = 2 \cdot n$ ☒ explicit ☐ recursive $a_3 =$ $2 \cdot 3 = 6$

15. $c_1 = 1; c_n = 2 \cdot c_{n-1}$ ☐ explicit ☒ recursive $c_3 =$ $2 \cdot c_2 = 2 \cdot 2 \cdot c_1 = 2 \cdot 2 \cdot 1 = 4$

16. $d_1 = 3; d_2 = 5; d_n = d_{n-1} + d_{n-2}$ ☐ explicit ☒ recursive $d_3 =$ $d_2 + d_1 = 3 + 5 = 8$

17. What advantage does an explicit formula have over a recursive formula? (3 points)

With explicit formulas, the n-th element can be calculated straight from the equation. This is a benefit over recursive since all n-1 previous elements must be calculated before the n-th element can be calculated.

18. True or False: Given the set A = {*ab*, *bc*, *ba*}, the string *abbabaab* belongs to the set ***A****. (2 points)

For problems 19, 20, and 21, tell whether or not the string on the left belongs to the regular set corresponding to the regular expression on the right. (2 points each)

19. string: 01111110 regular expression: 0(1*)0 ☒ Belongs ☐ Doesn't belong

The regular expression 0(1*)0 is any string from 1* = {Λ, 1, 11, 111, 1111, 11111, 111111, …} with a single zero in front of it and a single zero after it. Therefore, 01111110 belongs to the set generated by this regular expression.

20. string: Λ (empty string) regular expression: (*ab*)**c* ☐ Belongs ☒ Doesn't belong

The regular expression (*ab*)**c* is any string from (*ab*)* = {Λ, *ab*, *abab*, *ababab*, *abababab*, *ababababab*, …} followed by a single '*c*'. This requires every string to end in a '*c*', and since the empty string does not, Λ does not belong to the set generated by this regular expression.

21. string: *cccc* regular expression: (*a** ∨ *b*)*c** ☒ Belongs ☐ Doesn't belong

The regular expression (*a** ∨ *b*)*c** starts with the set defined by (*a** ∨ *b*). This equals ***either*** a selection from the set *a** = {Λ, *a*, *aa*, *aaa*, *aaaa*, *aaaaa*, …} **or** the single letter b. The last part of the string defined by the regular expression (*a** ∨ *b*)*c** is taken from the set defined by *c** = {Λ, *c*, *cc*, *ccc*, *cccc*, *ccccc*, …}. By taking the Λ from *a** and the *cccc* from *c**, we get *cccc*. Therefore, the string belongs to the set generated by this regular expression.

22. Write three elements that are members of the regular set corresponding to the regular expression 0(11)*1. (3 points)

The regular expression 0(11)*1 always starts with a 0 and ends with a 1. Between these two digits is any number of instances of a pair of 1's, i.e., any one of the sequences from the set (11)* = {Λ, 11, 1111, 111111, 11111111, 1111111111, …}. This gives us a set of elements beginning with a 0 followed by an odd number of 1's.

{01, 0111, 011111, 01111111, 0111111111, 011111111111, …}

23. Give the regular expression corresponding to the regular set {ad, abcd, abcbcd, abcbcbcd, abcbcbcbcd, abcbcbcbcbcd,…}. (3 points)

Examination of the set of elements shows that the sequences always start with *a* and end with *d*. This means that the regular expression will be *a*(something in the middle)*d*. The pattern in the middle begins as the empty set (Λ), then goes to repeating patterns of *bc*. Therefore, the "something in the middle" should be (*bc*)*. This gives us the answer:

$$a(bc)^*d$$

24. True or False: 53496_9 is a valid base 9 number. (2 points) 9 cannot be part of a base 9 value.

25. Write the expansion in base 5 of 221_{10}. (4 points)

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221 ÷ 5 = 44 with a remainder of 1
 44 ÷ 5 = 8  with a remainder of 4
  8 ÷ 5 = 1  with a remainder of 3
  1 ÷ 5 = 0  with a remainder of 1
```

Listing the remainders in reverse order gives us $\mathbf{1341_5}$.

26. What is the result of *f*(20) if *f* is the mod-3 function, i.e., calculate 20 mod 3. (2 points)

The mod function is basically a remainder function. For example, 20 mod 3 is the remainder after dividing 20 by 3. This gives us an answer of **2**.

27. Convert the base 5 number 322_5 to base 10. Just write out the formula with the correct values; do not worry about doing the final calculation. (3 points)

$3 \cdot 5^2 + 2 \cdot 5^1 + 2 \cdot 5^0 = 3 \cdot 25 + 2 \cdot 5 + 2 \cdot 1 = 75 + 10 + 2 = 87$

28. If $m = 24$ and $n = 6$, determine the values of q and r that satisfy the expression $m = q \cdot n + r$ such that $0 \leq r < n$. (2 points)

$24 = 4 \cdot 6 + 0$ q = 4 r = 0

29. Write the integer 6160_{10} as a product of powers of primes. (4 points)

$6160 \div 2 = 3080 \div 2 = 1540 \div 2 = 770 \div 2 = 385 \div 5 = 77 \div 7 = 11$

This shows us that there are 4 products of 2, 1 product of 5, 1 product of 7, and one product of 11. This is expanded to:

$6160 = 2^4 \cdot 5^1 \cdot 7^1 \cdot 11^1$

30. Use any method you wish to find the greatest common divisor of 84 and 770. (4 points)

There are a couple of ways to do this. First, you could break both 84 and 770 down to their multiples of prime numbers. This would give us:

$84 = 2^2 \cdot 3^1 \cdot 7^1$
$770 = 2^1 \cdot 5^1 \cdot 7^1 \cdot 11^1$

This shows us that 84 and 770 have common prime number divisors of one power each of 2 and 7. Multiplying these together gives us the GCD:

$$2 \cdot 7 = 14$$

Another way to do this is to do an iterative application of the principle of integers m = qn + r using m and n to hold the values we are trying to find for the GCD where m$\underline{>}$n.

$$770 = \underline{9} \cdot 84 + \underline{14}$$

$$84 = \underline{6} \cdot 14 + \underline{0}$$

The remainder of zero in the last expression shows that the n=14 is our GCD.

Answer: $\mathbf{14_{10}}$

31. Use any method you wish to find the least common multiple of 300 and 210. (4 points)

Once again, there are a couple of ways to do this. The first way involves breaking both 300 and 210 down to their multiples of prime numbers. This would give us:

$300 = 2^2 \cdot 3^1 \cdot 5^2$
$210 = 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1$

Finding the LCM uses these prime multiples differently than the process for finding GCD's does. In the case of the LCM, we need to take the highest power for each of the prime multiples and multiply them together for the LCM.

$$2^2 \cdot 3^1 \cdot 5^2 \cdot 7^1 = 4 \cdot 3 \cdot 25 \cdot 7 = 4 \cdot 25 \cdot 3 \cdot 7 = 100 \cdot 21 = 2100$$

Another way to do this is to find the GCD, then substitute it into the equation GCD(m,n)·LCM(m,n) = m·n.

$$300 = \underline{1} \cdot 210 + \underline{90}$$

$$210 = \underline{2} \cdot 90 + \underline{30}$$

$$90 = \underline{3} \cdot 30 + \underline{0}$$

The remainder of zero in the last expression shows that the n=30 is our GCD(300,210). Substituting it into the equation for the LCM gives us:

$$LCM(m,n) = m \cdot n \div GCD(m,n) = 300 \cdot 210 \div 30 = 63000 \cdot 30 = 2100$$

Answer: $\mathbf{2100_{10}}$

32. True or False: If $\boldsymbol{A}$ is a matrix, to compute $\boldsymbol{A}^2$, $\boldsymbol{A}$ must be a square matrix. (2 points)

33. True or False: If $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices, then $\boldsymbol{AB} = \boldsymbol{BA}$. (2 points)

For problems 34 through 37, use the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ shown below.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 4 & 2 \\ 3 & 1 \\ 2 & 5 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

34. True or False: It is possible to compute $\boldsymbol{A}^T + \boldsymbol{B}$. (2 points)

$$\boldsymbol{A}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 4 & 2 \\ 3 & 1 \\ 2 & 5 \end{bmatrix}$$

Since $\boldsymbol{A}^T$ and $\boldsymbol{B}$ do not have the same dimensions, they cannot be added. Therefore, the answer is FALSE.

35. True or False: The result of $\boldsymbol{BC}$ is matrix with 3 rows and 1 column. (2 points)

$$\boldsymbol{B} \cdot \boldsymbol{C} = \begin{bmatrix} 4 & 2 \\ 3 & 1 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \cdot 2 + 2 \cdot 3 \\ 3 \cdot 2 + 1 \cdot 3 \\ 2 \cdot 2 + 5 \cdot 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \\ 19 \end{bmatrix}$$

Since $\boldsymbol{B}$ has 3 rows and $\boldsymbol{C}$ has one column, the result of $\boldsymbol{B} \cdot \boldsymbol{C}$ is a matrix with 3 rows and 1 column. Therefore, the answer is TRUE.

36. True or False: It is possible to compute $\boldsymbol{CA} + \boldsymbol{B}^T$. (2 points)

$$\boldsymbol{C} \cdot \boldsymbol{A} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 2 & 2 \cdot 3 \\ 3 \cdot 1 & 3 \cdot 2 & 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Since $\boldsymbol{C} \cdot \boldsymbol{A}$ and $\boldsymbol{B}^T$ have the same dimensions, they can be added. Therefore, the answer is TRUE.

37. Calculate $\boldsymbol{AB}$. Show ***all*** work. (4 points)

$$\boldsymbol{A} \cdot \boldsymbol{B} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 \\ 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 3 + 3 \cdot 2 & 1 \cdot 2 + 2 \cdot 1 + 3 \cdot 5 \end{bmatrix} = \begin{bmatrix} 16 & 19 \end{bmatrix}$$