CS 373: Combinatorial Algorithms, Fall 2000

Homework 3 (due October 17, 2000 at midnight)

Name:		
Net ID:	Alias:	U $^3\!/_4$ 1

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Starting with Homework 1, homeworks may be done in teams of up to three people. Each team turns in just one solution, and every member of a team gets the same grade. Since 1-unit graduate students are required to solve problems that are worth extra credit for other students, **1-unit grad students may not be on the same team as 3/4-unit grad students or undergraduates.**

Neatly print your name(s), NetID(s), and the alias(es) you used for Homework 0 in the boxes above. Please also tell us whether you are an undergraduate, 3/4-unit grad student, or 1-unit grad student by circling U, $^3\!/_4$, or 1, respectively. Staple this sheet to the top of your homework.

Required Problems

1. Suppose you have to design a dictionary that holds 2048 items.

   (a) How many probes are used for an unsuccessful search if the dictionary is implemented as a sorted array? Assume the use of Binary Search.

   (b) How large a hashtable do you need if your goal is to have 2 as the expected number of probes for an unsuccessful search?

   (c) How much more space is needed by the hashtable compared to the sorted array? Assume that each pointer in a linked list takes 1 word of storage.

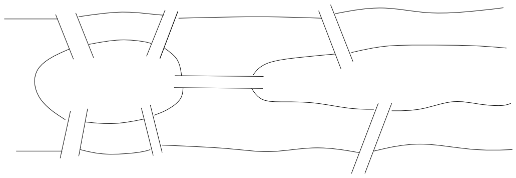
2. In order to facilitate recompiling programs from multiple source files when only a small number of files have been updated, there is a UNIX utility called 'make' that only recompiles those files that were changed after the most recent compilation, *and* any intermediate files in the compilation that depend on those that were changed. A Makefile is typically composed

of a list of source files that must be compiled. Each of these source files is dependent on some of the other files which are listed. Thus a source file must be recompiled if a file on which it depends is changed.

Assuming you have a list of which files have been recently changed, as well as a list for each source file of the files on which it depends, design an algorithm to recompile only those necessary. Don't worry about the details of parsing a Makefile.

3. A person wants to fly from city A to city B in the shortest possible time. She turns to the traveling agent who knows all the departure and arrival times of all the flights on the planet. Give an algorithm that will allow the agent to choose a route with the minimum total travel time—initial takeoff to final landing, including layovers. *[Hint: Modify the data and call a shortest-path algorithm.]*

4. During the eighteenth century the city of Königsberg in East Prussia was divided into four sections by the Pregel river. Seven bridges connected these regions, as shown below. It was said that residents spent their Sunday walks trying to find a way to walk about the city so as to cross each bridge exactly once and then return to their starting point.



   (a) Show how the residents of the city could accomplish such a walk or prove no such walk exists.

   (b) Given any undirected graph $G = (V, E)$, give an algorithm that finds a cycle in the graph that visits every edge exactly once, or says that it can't be done.

5. Suppose you have a graph G and an MST of that graph (i.e. the MST has already been constructed).

   (a) Give an algorithm to update the MST when an edge is added to G.

   (b) Give an algorithm to update the MST when an edge is deleted from G.

   (c) Give an algorithm to update the MST when a vertex (and possibly edges to it) is added to G.

6. *[This problem is required only for graduate students taking CS 373 for a full unit; anyone else can submit a solution for extra credit.]*

   You are given an unlimited number of each of n different types of envelopes. The dimensions of envelope type i are $x_i \times y_i$. In nesting envelopes inside one another, you can place envelope A inside envelope B if and only if the dimensions A are *strictly smaller* than the dimensions of B. Design and analyze an algorithm to determine the largest number of envelopes that can be nested inside one another.

Practice Problems

★1. Let the hash function for a table of size m be

$$h(x) = \lfloor Amx \rfloor \bmod m$$

where $A = \hat{\phi} = \frac{\sqrt{5}-1}{2}$. Show that this gives the best possible spread, i.e. if the x are hashed in order, $x+1$ will be hashed in the largest remaining contiguous interval.

2. The incidence matrix of an undirected graph $G = (V, E)$ is a $|V| \times |E|$ matrix $B = (b_{ij})$ such that

$$b_{ij} = \big[(i,j) \in E\big] = \begin{cases} 1 & \text{if } (i,j) \in E, \\ 0 & \text{if } (i,j) \notin E. \end{cases}$$

(a) Describe what all the entries of the matrix product BB^T represent (B^T is the matrix transpose).

(b) Describe what all the entries of the matrix product B^TB represent.

★(c) Let $C = BB^T - 2A$, where A is the adjacency matrix of G, with zeroes on the diagonal. Let C' be C with the first row and column removed. Show that $\det C'$ is the number of spanning trees.

3. (a) Give an $O(V)$ algorithm to decide whether a directed graph contains a *sink* in an adjacency matrix representation. A sink is a vertex with in-degree $V-1$.

(b) An undirected graph is a scorpion if it has a vertex of degree 1 (the sting) connected to a vertex of degree two (the tail) connected to a vertex of degree $V-2$ (the body) connected to the other $V-3$ vertices (the feet). Some of the feet may be connected to other feet.

Design an algorithm that decides whether a given adjacency matrix represents a scorpion by examining only $O(V)$ of the entries.

(c) Show that it is impossible to decide whether G has at least one edge in $O(V)$ time.

4. Given an *undirected* graph $G = (V, E)$, and a weight function $f : E \to \mathbb{R}$ on the *edges*, give an algorithm that finds (in time polynomial in V and E) a cycle of smallest weight in G.

5. Let $G = (V, E)$ be a graph with n vertices. A *simple path* of G, is a path that does not contain the same vertex twice. Use dynamic programming to design an algorithm (not polynomial time) to find a simple path of maximum length in G. Hint: It can be done in $O(n^c 2^n)$ time, for some constant c.

6. Suppose all edge weights in a graph G are equal. Give an algorithm to compute a minimum spanning tree of G.

7. Give an algorithm to construct a *transitive reduction* of a directed graph G, i.e. a graph G^{TR} with the fewest edges (but with the same vertices) such that there is a path from a to b in G iff there is also such a path in G^{TR}.

8. (a) What is $5^{2^{29^{5^0}+23^{4^1}+17^{3^2}+11^{2^3}+5^{1^4}}} \bmod 6$?

   (b) What is the capital of Nebraska? Hint: It is not Omaha. It is named after a famous president of the United States that was not George Washington. The distance from the Earth to the Moon averages roughly 384,000 km.