Blech! Ack! Oop! THPPFFT!
— Bill the Cat, "Bloom County" (1980)

14 Fast Fourier Transforms (November 7)

14.1 Polynomials

In this lecture we'll talk about algorithms for manipulating *polynomials*: functions of one variable built from additions subtractions, and multiplications (but no divisions). The most common representation for a polynomial $p(x)$ is as a sum of weighted powers of a variable x:

$$\boxed{p(x) = \sum_{j=0}^{n} a_j x^j.}$$

The numbers a_j are called *coefficients*. The *degree* of the polynomial is the largest power of x; in the example above, the degree is n. Any polynomial of degree n can be specified by a sequence of $n+1$ coefficients. Some of these coefficients may be zero, but not the nth coefficient, because otherwise the degree would be less than n.

Here are three of the most common operations that are performed with polynomials:

- **Evaluate:** Give a polynomial p and a number x, compute the number $p(x)$.
- **Add:** Give two polynomials p and q, compute a polynomial $r = p + q$, so that $r(x) = p(x) + q(x)$ for all x. If p and q both have degree n, then their sum $p + q$ also has degree n.
- **Multiply:** Give two polynomials p and q, compute a polynomial $r = p \cdot q$, so that $r(x) = p(x) \cdot q(x)$ for all x. If p and q both have degree n, then their product $p \cdot q$ has degree $2n$.

Suppose we represent a polynomial of degree n as an array of $n+1$ coefficients $P[0\,..\,n]$, where $P[j]$ is the coefficient of the x^j term. We learned simple algorithms for all three of these operations in high-school algebra:

```
EVALUATE($P[0..n], x$):
  $X \gets 1$    ⟨⟨$X = x^j$⟩⟩
  $y \gets 0$
  for $j \gets 0$ to $n$
      $y \gets y + P[j] \cdot X$
      $X \gets X \cdot x$
  return $y$
```

```
ADD($P[0..n], Q[0..n]$):
  for $j \gets 0$ to $n$
      $R[j] \gets P[j] + Q[j]$
  return $R[0..n]$
```

```
MULTIPLY($P[0..n], Q[0..m]$):
  for $j \gets 0$ to $n + m$
      $R[j] \gets 0$
  for $j \gets 0$ to $n$
      for $k \gets 0$ to $m$
          $R[j+k] \gets P[j] \cdot Q[k]$
  return $R[0..n+m]$
```

EVALUATE uses $O(n)$ arithmetic operations.[1] This is the best you can do in theory, but we can cut the number of multiplications in half using *Horner's rule*:

$$p(x) = a_0 + x(a_1 + x(a_2 + \ldots + xa_n)).$$

[1] I'm going to assume in this lecture that each arithmetic operation takes $O(1)$ time. This may not be true in practice; in fact, one of the most powerful applications of FFTs is fast *integer* multiplication. One of the fastest integer multiplication algorithms, due to Schönhage and Strassen, multiplies two n-bit binary numbers in $O(n \log n \log\log n \log\log\log n \log\log\log\log n \cdots)$ time. The algorithm uses an n-element Fast Fourier Transform, which requires several $O(\log n)$-nit integer multiplications. These smaller multiplications are carried out recursively (of course!), which leads to the cascade of logs in the running time. Needless to say, this is a can of worms.

```
HORNER($P[0\,..\,n], x$):
    $y \leftarrow P[n]$
    for $i \leftarrow n-1$ downto 0
        $y \leftarrow x \cdot y + P[i]$
    return $y$
```

The addition algorithm also runs in $O(n)$ time, and this is clearly the best we can do.

The multiplication algorithm, however, runs in $O(n^2)$ time. In the very first lecture, we saw a divide and conquer algorithm (due to Karatsuba) for multiplying two n-bit integers in only $O(n^{\lg 3})$ steps; precisely the same algorithm can be applied here. Even cleverer divide-and-conquer strategies lead to multiplication algorithms whose running times are arbitrarily close to linear—$O(n^{1+\varepsilon})$ for your favorite value $e > 0$—but with great cleverness comes great confusion. These algorithms are difficult to understand, even more difficult to implement correctly, and not worth the trouble in practice thanks to large constant factors.

14.2 Alternate Representations: Roots and Samples

Part of what makes multiplication so much harder than the other two operations is our input representation. Coefficients vectors are the most common representation for polynomials, but there are at least two other useful representations.

The first exploits the fundamental theorem of algebra: Every polynomial p of degree n has n *roots* $r_1, r_2, \ldots r_n$ such that $p(r_j) = 0$ for all j. Some of these roots may be irrational; some of these roots may by complex; and some of these roots may be repeated. Despite these complications, we do get a unique representation of any polynomial of the form

$$\boxed{p(x) = s\prod_{j=1}^{n}(x - r_j)}$$

where the r_j's are the roots and s is a scale factor. Once again, to represent a polynomial of degree n, we need a list of $n+1$ numbers: one scale factor and n roots.

Given a polynomial in root representation, we can clearly evaluate it in $O(n)$ time. Given two polynomials in root representation, we can easily multiply them in $O(n)$ time by multiplying their scale factors and just concatenating the two root sequences; in fact, if we don't care about keeping the old polynomials around, we can compute their product in $O(1)$ time! Unfortunately if we want to *add* two polynomials in root representation, we're pretty much out of luck; there's essentially no correlation between the roots of p, the roots of q, and the roots of $p + q$. We could convert the polynomials to the more familiar coefficient representation first—this takes $O(n^2)$ time using the high-school algorithms—but there's no easy way to convert the answer back. In fact, given a polynomial in coefficient form, it's usually *impossible* to compute its roots exactly.[2]

Our third representation for polynomials comes from the following consequence of the fundamental theorem of algebra. Given a list of $n + 1$ pairs $\{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\}$, there is *exactly one* polynomial p of degree n such that $p(x_j) = y_j$ for all j. This is just a generalization of the fact that any two points determine a unique line, since a line is (the graph of) a polynomial of degree 1. We say that the polynomial p *interpolates* the points (x_j, y_j). As long as we agree on the sample locations x_j in advance, we once again need exactly $n + 1$ numbers to represent a polynomial of degree n.

[2] This is where numerical analysis comes from.

Adding or multiplying two polynomials in this sample representation is easy, as long as they use the same sample locations x_j. To add the polynomials, just add their sample values. To multiply two polynomials, just multiply their sample values; however, if we're multiplying two polynomials of degree n, we need to *start* with $2n+1$ sample values for each polynomial, since that's how many we need to uniquely represent the product polynomial. Both algorithms run in $O(n)$ time.

Unfortunately, evaluating a polynomial in this representation is no longer trivial. The following formula, due to Lagrange, allows us to compute the value of any polynomial of degree n at any point, given a set of $n+1$ samples.

$$p(x) = \sum_{j=0}^{n-1} \left(y_j \frac{\prod_{k\neq j}(x - x_k)}{\prod_{k\neq j}(x_j - x_k)} \right) = \sum_{j=0}^{n-1} \left(\frac{y_j}{\prod_{k\neq j}(x_j - x_k)} \prod_{k\neq j}(x - x_k) \right)$$

Hopefully it's clear that formula actually describes a polynomial, since each term in the rightmost sum is written as a scaled product of monomials. It's also not hard to check that $p(x_j) = y_j$ for all j. As I mentioned earlier, the fact that this is *the only* polynomial that interpolates the points $\{(x_j, y_j)\}$ is an easy consequence of the fundamental theorem of algebra. We can easily transform this formula into an $O(n^2)$-time algorithm.

We find ourselves in th following frustrating situation. We have three representations for polynomials and three basic operations. Each representation allows us to almost trivially perform a different pair of operations in linear time, but the third takes at least quadratic time, if it can be done at all!

	evaluate	add	multiply
coefficients	$O(n)$	$O(n)$	$O(n^2)$
roots + scale	$O(n)$	∞	$O(n)$
samples	$O(n^2)$	$O(n)$	$O(n)$

14.3 Converting Between Representations?

What we need are fast algorithms to convert quickly from one representation to another. That way, when we need to perform an operation that's hard for our default representation, we can switch to a different representation that makes the operation easy, perform that operation, and then switch back. This strategy immediately rules out the root representation, since (as I mentioned earlier) finding roots of polynomials is impossible in general, at least if we're interested in exact results.

So how do we convert from coefficients to samples and back? Clearly, once we choose our sample positions x_j, we can compute each sample value $y_j = p(x_j)$ in $O(n)$ time from the coefficients. So we can convert a polynomial of degree n from coefficients to samples in $O(n^2)$ time. The Lagrange formula gives us an explicit conversion algorithm from the sample representation back to the more familiar coefficient representation. If we use the naïve algorithms for adding and multiplying polynomials (in coefficient form), this conversion takes $O(n^3)$ time.

This looks pretty bad, until we realize there's a degree of freedom we haven't exploited yet. ***Whenever we convert from coefficients to samples, we get to choose the sample points!*** Our slow algorithms may be slow only because we're trying to be too general. Perhaps, if we choose a set of sample points with just the right kind of recursive structure, we can do the conversion more quickly. In fact, there is a set of sample points that's perfect for the job.

14.4 The Discrete Fourier Transform

Given a polynomial of degree $n-1$, we'd like to find n sample points that are somehow as symmetric as possible. The most natural choice for those n points are the *nth roots of unity*; these are the roots of the polynomial $x^n - 1 = 0$. These n roots are spaced exactly evenly around the unit circle in the complex plane.[3] Every nth root of unity is a power of the *primitive* root

$$\boxed{\omega_n = e^{2\pi i/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.}$$

A typical nth root of unity has the form

$$\omega_n^j = e^{(2\pi i/n)j} = \cos\left(\frac{2\pi}{n}j\right) + i\sin\left(\frac{2\pi}{n}j\right).$$

These complex numbers have several useful properties for any integers n and k:

- There are only n different nth roots of unity: $\omega_n^k = \omega_n^{k \bmod n}$.
- If n is even, then $\omega_n^{k+n/2} = -\omega_n^k$; in particular, $\omega_n^{n/2} = -\omega_n^0 = -1$.
- $1/\omega_n^k = \omega_n^{-k} = \overline{\omega_n^k} = (\overline{\omega_n})^k$, where the bar represents complex conjugation: $\overline{a+bi} = a - bi$
- $\omega_n = \omega_{kn}^k$. Thus, every nth root of unity is also a (kn)th root of unity.

If we sample a polynomial of degree $n-1$ at the nth roots of unity, the resulting list of sample values is called the *discrete Fourier transform* of the polynomial (or more formally, of the coefficient vector). Thus, given an array $P[0\,..\,n-1]$ of coefficients, the discrete Fourier transform computes a new vector $P^*[0\,..\,n-1]$ where

$$\boxed{\boxed{P^*[j] = p(\omega_n^j) = \sum_{k=0}^{n-1} P[k]\cdot\omega_n^{jk}}}$$

We can obviously compute P^* in $O(n^2)$ time, but the structure of the nth roots of unity lets us do better. But before we describe that faster algorithm, let's think about how we might invert this transformation.

It's not hard to see that the discrete Fourier transform—in fact, any conversion from a vector of coefficients to a vector of sample values—is a *linear* transformation. The DFT just multiplies the coefficient vector by a matrix V to obtain the sample vector. Each entry in V is an nth root of unity; specifically, $\boxed{v_{jk} = \omega_n^{jk}}$ for all j, k.

$$V = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \cdots & \omega_n^{2(n-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \cdots & \omega_n^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \omega_n^{3(n-1)} & \cdots & \omega_n^{(n-1)^2} \end{bmatrix}$$

[3] In this lecture, i always represents the square root of -1. Most computer scientists are used to thinking of i as an integer index into a sequence, an array, or a for-loop, but we obviously can't do that here. The physicist's habit of using $j = \sqrt{-1}$ just delays the problem (how do physicists write quaternions?), and typographical tricks like I or $\mathbf{i}$ or Mathematica's ⅈ are just stupid.

To invert the discrete Fourier transform, we just have to multiply P^* by the inverse matrix V^{-1}. But this is almost the same as multiplying by V itself, because of the following amazing fact:

$$\boxed{V^{-1} = \overline{V}/n}$$

In other words, if $W = V^{-1}$ then $w_{jk} = \overline{v_{jk}}/n = \overline{\omega_n^{jk}}/n = \omega_n^{-jk}/n$. It's not hard to prove this fact with a little linear algebra.

Proof: We just have to show that $M = VW$ is the identity matrix. We can compute a single entry in this matrix as follows:

$$m_{jk} = \sum_{l=0}^{n-1} v_{jl} \cdot w_{lk} = \sum_{l=0}^{n-1} \omega_n^{jl} \cdot \overline{\omega_n}^{lk}/n = \frac{1}{n}\sum_{l=0}^{n-1} \omega_n^{jl-lk} = \frac{1}{n}\sum_{l=0}^{n-1} (\omega_n^{j-k})^l$$

If $j = k$, then $\omega_n^{j-k} = 1$, so

$$m_{jk} = \frac{1}{n}\sum_{l=0}^{n-1} 1 = \frac{n}{n} = 1,$$

and if $j \neq k$, we have a geometric series

$$m_{jk} = \sum_{l=0}^{n-1} (\omega_n^{j-k})^l = \frac{(\omega_n^{j-k})^n - 1}{\omega_n^{j-k} - 1} = \frac{(\omega_n^n)^{j-k} - 1}{\omega_n^{j-k} - 1} = \frac{1^{j-k} - 1}{\omega_n^{j-k} - 1} = 0.$$

That's it! □

What this means for us computer scientists is that any algorithm for computing the discrete Fourier transform can be easily modified to compute the inverse transform as well.

14.5 Divide and Conquer

The structure of the matrix V also allows us to compute the discrete Fourier transform efficiently using a divide and conquer strategy. The basic structure of the algorithm is almost the same as MergeSort, and the $O(n \log n)$ running time will ultimately follow from the same recurrence. The *Fast Fourier Transform* algorithm, popularized by Cooley and Tukey in 1965[4], assumes that n is a power of two; if necessary, we can just pad the coefficient vector with zeros.

To get an idea of how the divide-and-conquer strategy works, let's look at the DFT matrixes for $n = 8$. To simplify notation, let $\omega = \omega_8 = \sqrt{2}/2 + i\sqrt{2}/2$.

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\
1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\
1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\
1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\
1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\
1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\
1 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49}
\end{bmatrix}
=
\begin{bmatrix}
\boxed{1} & 1 & \boxed{1} & 1 & \boxed{1} & 1 & \boxed{1} & 1 \\
\boxed{1} & \omega & \boxed{i} & \overline{\omega} & \boxed{-1} & -\omega & \boxed{-i} & -\overline{\omega} \\
\boxed{1} & i & \boxed{-1} & -i & \boxed{1} & i & \boxed{-1} & -i \\
\boxed{1} & \overline{\omega} & \boxed{i} & \omega & \boxed{-1} & -\overline{\omega} & \boxed{-i} & -\omega \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -\omega & i & -\overline{\omega} & -1 & \omega & -i & \overline{\omega} \\
1 & -i & -1 & i & 1 & -i & -1 & i \\
1 & -\overline{\omega} & -i & -\omega & -1 & \overline{\omega} & i & \omega
\end{bmatrix}$$

[4] Actually, the FFT algorithm was previously published by Runge and König in 1924, and again by Yates in 1932, and again by Stumpf in 1937, and again by Danielson and Lanczos in 1942. But it was first *used* by Gauss in the 1800s for calculating the paths of asteroids from a finite number of equally-spaced observations. By hand. Fourier always did it the hard way. Cooley and Tukey apparently developed their algorithm to help detect Soviet nuclear tests without actually visiting Soviet nuclear facilities, by interpolating off-shore seismic readings. Without their rediscovery of the FFT algorithm, the nuclear test ban treaty would never have been ratified, and we'd all be speaking Russian, or more likely, whatever language radioactive glass speaks.

The boxed entries actually form the DFT matrix for $n = 4$!

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4 & \omega_4^2 & \omega_4^3 \\ 1 & \omega_4^2 & \omega_4^4 & \omega_4^6 \\ 1 & \omega_4^3 & \omega_4^6 & \omega_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

The input to the FFT algorithm is an array $P[0\,..\,n-1]$ of coefficients of a polynomial $p(x)$ with degree $n-1$. We start by splitting p into two smaller polynomials u and v, each with degree $n/2-1$, by setting

$$U[k] = P[2k] \qquad \text{and} \qquad V[k] = P[2k-1].$$

In other words, u has all the even-degree coefficients of p, and v has all the odd-degree coefficients. For example, if $p(x) = 3x^3 - 4x^2 + 7x + 5$, then $u(x) = -4x + 5$ and $v(x) = 3x + 7$. These three polynomials satisfy the equation

$$p(x) = u(x^2) + x \cdot v(x^2).$$

In particular, if x is an nth root of unity, we have

$$P^*[k] = p(\omega_n^k) = u(\omega_n^{2k}) + \omega_n^k \cdot v(\omega_n^{2k}).$$

Now we can exploit those roots of unity again. Since n is a power of two, n must be even, so we have $\omega_n^{2k} = \omega_{n/2}^k = \omega_{n/2}^{k \bmod n/2}$. In other words, the values of p at the nth roots of unity depend on the values of u and v at $(n/2)$th roots of unity. Those are just coefficients in the DFTs of u and v!

$$P^*[k] = U^*[k \bmod n/2] + \omega_n^k \cdot V^*[k \bmod n/2]$$

So once we recursively compute U^* and V^*, we can compute P^* in linear time. The base case for the recurrence is $n = 1$. The overall running time satisfies the recurrence $T(n) = \Theta(n) + 2T(n/2)$, which as we all know solves to $T(n) = \Theta(n \log n)$.

Here's the complete FFT algorithm, along with its inverse.

```
FFT($P[0\,..\,n-1]$):
    if $n = 1$
        return $P$

    for $j \leftarrow 0$ to $n/2 - 1$
        $U[j] \leftarrow P[2j]$
        $V[j] \leftarrow P[2j+1]$

    $U^* \leftarrow$ FFT($U[0\,..\,n/2-1]$)
    $V^* \leftarrow$ FFT($V[0\,..\,n/2-1]$)

    $\omega_n \leftarrow \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$
    $\omega \leftarrow 1$

    for $j \leftarrow 0$ to $n/2 - 1$
        $P^*[j]       \leftarrow U^*[j] + \omega \cdot V^*[j]$
        $P^*[j + n/2] \leftarrow U^*[j] - \omega \cdot V^*[j]$
        $\omega \leftarrow \omega \cdot \omega_n$

    return $P^*[0\,..\,n-1]$
```

```
InverseFFT($P^*[0\,..\,n-1]$):
    if $n = 1$
        return $P$

    for $j \leftarrow 0$ to $n/2 - 1$
        $U^*[j] \leftarrow P^*[2j]$
        $V^*[j] \leftarrow P^*[2j+1]$

    $U \leftarrow$ InverseFFT($U[0\,..\,n/2-1]$)
    $V \leftarrow$ InverseFFT($V[0\,..\,n/2-1]$)

    $\overline{\omega_n} \leftarrow \cos(\frac{2\pi}{n}) - i\sin(\frac{2\pi}{n})$
    $\omega \leftarrow 1$

    for $j \leftarrow 0$ to $n/2 - 1$
        $P[j]       \leftarrow 2(U[j] + \omega \cdot V[j])$
        $P[j + n/2] \leftarrow 2(U[j] - \omega \cdot V[j])$
        $\omega \leftarrow \omega \cdot \overline{\omega_n}$

    return $P[0\,..\,n-1]$
```

Given two polynomials p and q, each represented by an array of coefficients, we can multiply them in $\Theta(n \log n)$ arithmetic operations as follows. First, pad the coefficient vectors and with zeros until the size is a power of two greater than or equal to the sum of the degrees. Then compute the DFTs of each coefficient vector, multiply the sample values one by one, and compute the inverse DFT of the resulting sample vector.

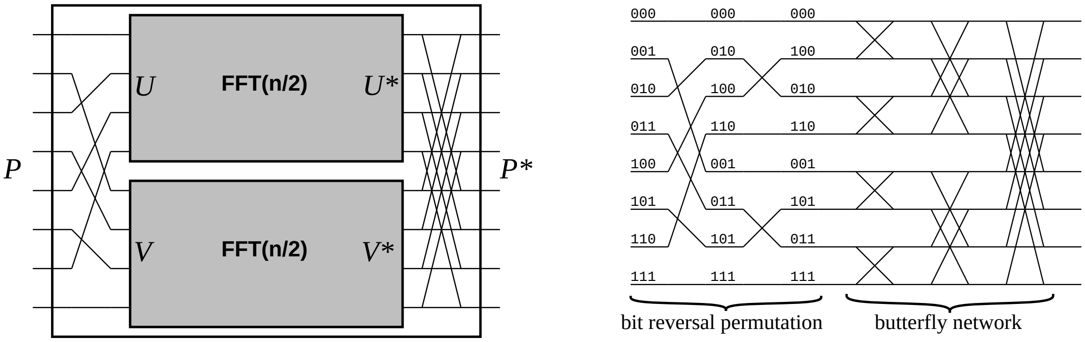
```
FFTMULTIPLY($P[0..n-1], Q[0..m-1]$):
  $\ell \leftarrow \lceil \lg(n+m) \rceil$
  for $j \leftarrow n$ to $2^\ell - 1$
      $P[j] \leftarrow 0$
  for $j \leftarrow m$ to $2^\ell - 1$
      $Q[j] \leftarrow 0$

  $P^* \leftarrow FFT(P)$
  $Q^* \leftarrow FFT(Q)$
  for $j \leftarrow 0$ to $2^\ell - 1$
      $R^*[j] \leftarrow P^*[j] \cdot Q^*[j]$

  return INVERSEFFT($R^*$)
```

14.6 Inside the FFT

FFTs are often implemented in hardware as circuits. To see the recursive structure of the circuit, let's connect the top-level inputs and outputs to the inputs and outputs of the recursive calls. On the left we split the input P into two recursive inputs U and V. On the right, we compbine the outputs U^* and V^* to obtain the final output P^*.



The recursive structure of the FFT algorithm.

If we expand this recursive structure completely, we see that the circuit splits naturally into two parts. The left half computes the *bit-reversal permuation* of the input. To find the position of $P[k]$ in this permutation, write k in binary, adn then read the bits backwards. For example, in an 8-element bit-reversal permutation, $P[3] = P[011_2]$ ends up in position $6 = 110_2$. The right half of the FFT circuit is a *butterlfy network*. Butterfly networks are often used to route between processors in massively-parallel computers, since they allow any processor to communicate with any other in only $O(\log n)$ steps.