12 The Cycle Space and Bond Space

12.1 CIRCULATIONS AND POTENTIAL DIFFERENCES

Let D be a digraph. A real-valued function f on A is called a *circulation* in D if it satisfies the conservation condition at each vertex:

$$f^-(v) = f^+(v) \quad \text{for all} \quad v \in V \tag{12.1}$$

If we think of D as an electrical network, then such a function f represents a circulation of currents in D. Figure 12.1 shows a circulation in a digraph.

If f and g are any two circulations and r is any real number, then it is easy to verify that both $f+g$ and rf are also circulations. Thus the set of all circulations in D is a vector space. We denote this space by $\mathscr{C}$. In what follows, we shall find it convenient to identify a subset S of A with $D[S]$, the subdigraph of D induced by S.

There are certain circulations of special interest. These are associated with cycles in D. Let C be a cycle in D with an assigned orientation and let C^+ denote the set of arcs of C whose direction agrees with this orientation. We associate with C the function f_C defined by

$$f_C(a) = \begin{cases} 1 & \text{if} \quad a \in C^+ \\ -1 & \text{if} \quad a \in C \setminus C^+ \\ 0 & \text{if} \quad a \notin C \end{cases}$$

Clearly, f_C satisfies (12.1) and hence is a circulation. Figure 12.2 depicts a circulation associated with a cycle.

We shall see later on that each circulation is a linear combination of the circulations associated with cycles. For this reason we refer to $\mathscr{C}$ as the *cycle space* of D.

We now turn our attention to a related class of functions. Given a function p on the vertex set V of D, we define the function δp on the arc set A by the rule that, if an arc a has tail x and head y, then

$$\delta p(a) = p(x) - p(y) \tag{12.2}$$

If D is thought of as an electrical network with potential $p(v)$ at v, then, by (12.2), δp represents the potential difference along the wires of the network. For this reason a function g on A is called a *potential difference* in D if

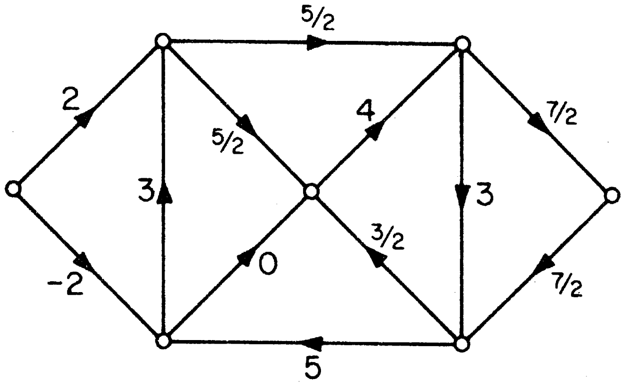


Figure 12.1. A circulation

$g = \delta p$ for some function p on V. Figure 12.3 shows a digraph with an assignment of potentials to its vertices and the corresponding potential difference.

As with circulations, the set $\mathscr{B}$ of all potential differences in D is closed under addition and scalar multiplication and, hence, is a vector space.

Analogous to the function f_C associated with a cycle C, there is a function g_B associated with a bond B. Let $B = [S, \bar{S}]$ be a bond of D. We define g_B by

$$g_B(a) = \begin{cases} 1 & \text{if} \quad a \in (S, \bar{S}) \\ -1 & \text{if} \quad a \in (\bar{S}, S) \\ 0 & \text{if} \quad a \notin B \end{cases}$$

It can be verified that $g_B = \delta p$ where

$$p(v) = \begin{cases} 1 & \text{if} \quad v \in S \\ 0 & \text{if} \quad v \in \bar{S} \end{cases}$$

Figure 12.4 depicts the potential difference associated with a bond.

We shall see that each potential difference is a linear combination of potential differences associated with bonds. For this reason we refer to $\mathscr{B}$ as the *bond space* of D.

In studying the properties of the two vector spaces $\mathscr{B}$ and $\mathscr{C}$, we shall find

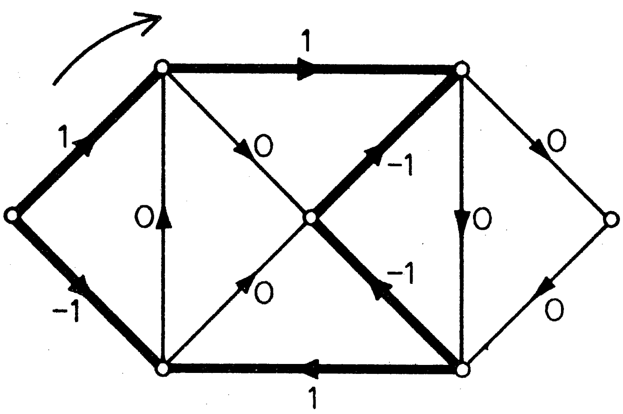


Figure 12.2

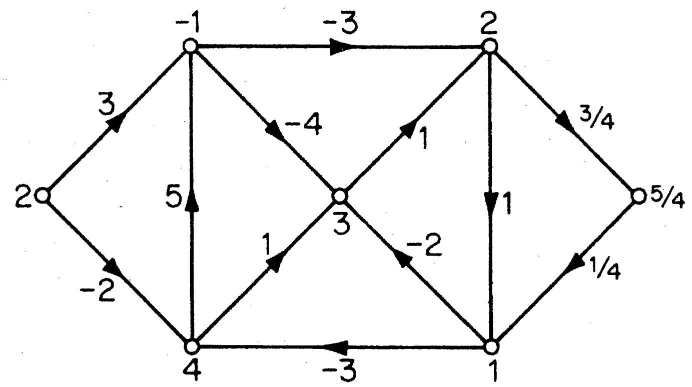


Figure 12.3. A potential difference

it convenient to regard a function on A as a row vector whose coordinates are labelled with the elements of A. The relationship between $\mathcal{B}$ and $\mathcal{C}$ is best seen by introducing the incidence matrix of D. With each vertex v of D we associate the function m_v on A defined by

$$m_v(a) = \begin{cases} 1 & \text{if } a \text{ is a link and } v \text{ is the tail of } a \\ -1 & \text{if } a \text{ is a link and } v \text{ is the head of } a \\ 0 & \text{otherwise} \end{cases}$$

The *incidence matrix* of D is the matrix $\boldsymbol{M}$ whose rows are the functions m_v. Figure 12.5 shows a digraph and its incidence matrix.

Theorem 12.1 Let $\boldsymbol{M}$ be the incidence matrix of a digraph D. Then $\mathcal{B}$ is the row space of $\boldsymbol{M}$ and $\mathcal{C}$ is its orthogonal complement.

Proof Let $g = \delta p$ be a potential difference in D. It follows from (12.2) that

$$g(a) = \sum_{v \in V} p(v)m_v(a) \quad \text{for all} \quad a \in A$$

Thus g is a linear combination of the rows of $\boldsymbol{M}$. Conversely, any linear combination of the rows of $\boldsymbol{M}$ is a potential difference. Hence $\mathcal{B}$ is the row space of $\boldsymbol{M}$.

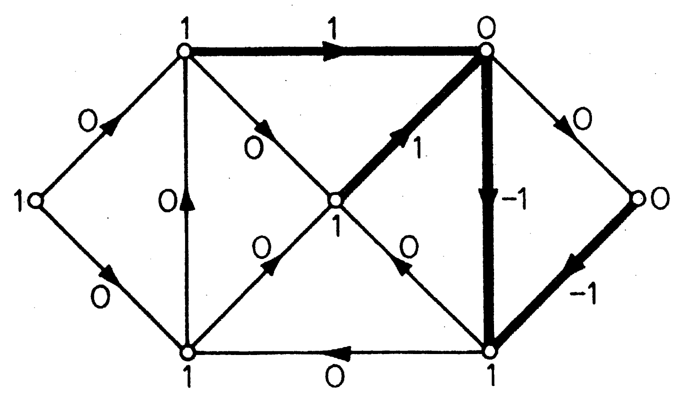
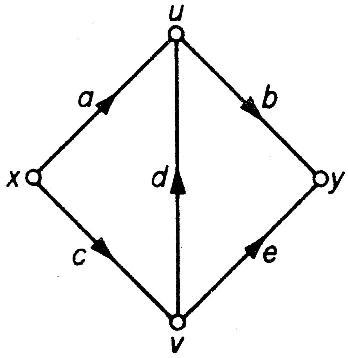


Figure 12.4



(a)

	a	b	c	d	e
x	1	0	1	0	0
u	-1	1	0	-1	0
v	0	0	-1	1	1
y	0	-1	0	0	-1

(b)

Figure 12.5. (*a*) D; (*b*) the incidence matrix of D

Now let f be a function on A. The condition (12.1) for f to be a circulation can be rewritten as

$$\sum_{a\in A} m_v(a)f(a)=0 \quad \text{for all} \quad v\in V$$

This implies that f is a circulation if and only if it is orthogonal to each row of $\mathbf{M}$. Hence $\mathscr{C}$ is the orthogonal complement of $\mathscr{B}$ □

The *support* of a function f on A is the set of elements of A at which the value of f is nonzero. We denote the support of f by $\|f\|$.

Lemma 12.2.1 If f is a nonzero circulation, then $\|f\|$ contains a cycle.

Proof This follows immediately, since $\|f\|$ clearly cannot contain a vertex of degree one □

Lemma 12.2.2 If g is a nonzero potential difference, then $\|g\|$ contains a bond.

Proof Let $g=\delta p$ be a nonzero potential difference in D. Choose a vertex $u\in V$ which is incident with an arc of $\|g\|$ and set

$$U=\{v\in V \mid p(v)=p(u)\}$$

Clearly, $\|g\|\supseteq[U, \bar{U}]$ since $g(a)\neq 0$ for all $a\in[U, \bar{U}]$. But, by the choice of u, $[U, \bar{U}]$ is nonempty. Thus $\|g\|$ contains a bond □

A matrix $\mathbf{B}$ is called a *basis matrix* of $\mathscr{B}$ if the rows of $\mathbf{B}$ form a basis for $\mathscr{B}$; a basis matrix of $\mathscr{C}$ is similarly defined. We shall find the following notation convenient. If $\mathbf{R}$ is a matrix whose columns are labelled with the elements of A, and if $S\subseteq A$, we shall denote by $\mathbf{R}\mid S$ the submatrix of $\mathbf{R}$ consisting of those columns of $\mathbf{R}$ labelled with elements in S. If $\mathbf{R}$ has a single row, our notation is the same as the usual notation for the restriction of a function to a subset of its domain.

Theorem 12.2 Let **B** and **C** be basis matrices of $\mathscr{B}$ and $\mathscr{C}$, respectively. Then, for any $S \subseteq A$

(i) the columns of $\mathbf{B} \mid S$ are linearly independent if and only if S is acyclic, and
(ii) the columns of $\mathbf{C} \mid S$ are linearly independent if and only if S contains no bond.

Proof Denote the column of **B** corresponding to arc a by $\mathbf{B}(a)$. The columns of $\mathbf{B} \mid S$ are linearly dependent if and only if there exists a function f on A such that

$$f(a) \neq 0 \quad \text{for some} \quad a \in S$$

$$f(a) = 0 \quad \text{for all} \quad a \notin S$$

and

$$\sum_{a \in A} f(a)\mathbf{B}(a) = \mathbf{O}$$

We conclude that the columns of $\mathbf{B} \mid S$ are linearly dependent if and only if there exists a nonzero circulation f such that $\|f\| \subseteq S$. Now if there is such an f then, by lemma 12.2.1, S contains a cycle. On the other hand, if S contains a cycle C, then f_C is a nonzero circulation with $\|f_C\| = C \subseteq S$. It follows that the columns of $\mathbf{B} \mid S$ are linearly independent if and only if S is acyclic. A similar argument using lemma 12.2.2 yields a proof of (ii) □

Corollary 12.2 The dimensions of $\mathscr{B}$ and $\mathscr{C}$ are given by

$$\dim \mathscr{B} = \nu - \omega \tag{12.3}$$

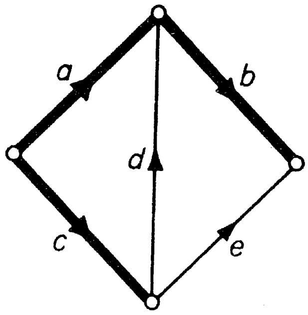
$$\dim \mathscr{C} = \varepsilon - \nu + \omega \tag{12.4}$$

Proof Consider a basis matrix **B** of $\mathscr{B}$. By theorem 12.2

$$\text{rank } \mathbf{B} = \max\{|S| \mid S \subseteq A, S \text{ acyclic}\}$$

The above maximum is attained when S is a maximal forest of D, and is therefore (exercise 2.2.4) equal to $\nu - \omega$. Since $\dim \mathscr{B} = \text{rank } \mathbf{B}$, this establishes (12.3). Now (12.4) follows, since $\mathscr{C}$ is the orthogonal complement of $\mathscr{B}$ □

Let T be a maximal forest of D. Associated with T is a special basis matrix of $\mathscr{C}$. If a is an arc of $\bar{T}$, then $T + a$ contains a unique cycle. Let C_a denote this cycle and let f_a denote the circulation corresponding to C_a, defined so that $f_a(a) = 1$. The $(\varepsilon - \nu + \omega) \times \varepsilon$ matrix **C** whose rows are f_a, $a \in \bar{T}$, is a basis matrix of $\mathscr{C}$. This follows from the fact that each row is a circulation and that rank $\mathbf{C} = \varepsilon - \nu + \omega$ (because $\mathbf{C} \mid \bar{T}$ is an identity matrix). We refer to **C** as the basis matrix of $\mathscr{C}$ *corresponding to T*. Figure 12.6*b* shows the basis matrix of $\mathscr{C}$ corresponding to the tree indicated in figure 12.6*a*.



(a)

	a	b	c	d	e
f_d	−1	0	1	1	0
f_e	−1	−1	1	0	1

(b)

	a	b	c	d	e
g_a	1	0	0	1	1
g_b	0	1	0	0	1
g_c	0	0	1	−1	−1

(c)

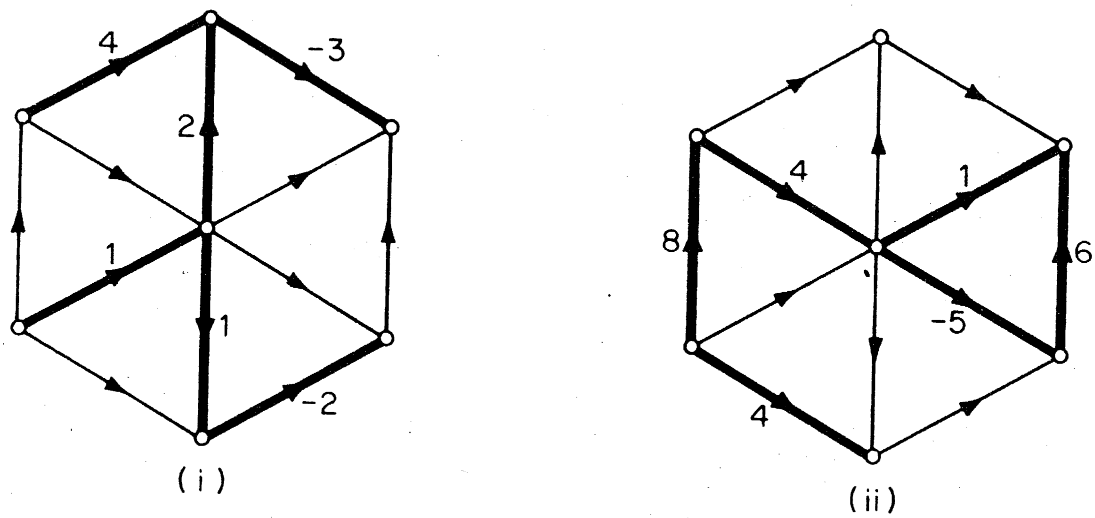
Figure 12.6

Analogously, if a is an arc of T, then $\bar{T}+a$ contains a unique bond (see theorem 2.6). Let B_a denote this bond and g_a the potential difference corresponding to B_a, defined so that $g_a(a)=1$. The $(\nu-\omega)\times\varepsilon$ matrix $\mathbf{B}$ whose rows are g_a, $a\in T$, is a basis matrix of $\mathscr{B}$, called the basis matrix of $\mathscr{B}$ *corresponding to T*. Figure 12.6*c* gives an example of such a matrix.

The relationship between cycles and bonds that has become apparent from the foregoing discussion finds its proper setting in the theory of matroids. The interested reader is referred to Tutte (1971).

Exercises

12.1.1 (*a*) In figure (i) below is indicated a function on a spanning tree and in figure (ii) a function on the complement of the tree. Extend the function in (i) to a potential difference and the function in (ii) to a circulation.



(*b*) Let f be a circulation and g a potential difference in D, and let T be a spanning tree of D. Show that f is uniquely determined by $f\mid\bar{T}$ and g by $g\mid T$.

12.1.2 (*a*) Let $\mathbf{B}$ and $\mathbf{C}$ be basis matrices of $\mathscr{B}$ and $\mathscr{C}$ and let T be any spanning tree of D. Show that $\mathbf{B}$ is uniquely determined by $\mathbf{B}\mid T$ and $\mathbf{C}$ is uniquely determined by $\mathbf{C}\mid\bar{T}$.

(*b*) Let T and T_1 be two fixed spanning trees of D. Let $\mathbf{B}$ and $\mathbf{B}_1$ denote the basis matrices of $\mathscr{B}$, and $\mathbf{C}$ and $\mathbf{C}_1$ the basis matrices of $\mathscr{C}$, corresponding to the trees T and T_1. Show that $\mathbf{B}=(\mathbf{B}\mid T_1)\mathbf{B}_1$ and $\mathbf{C}=(\mathbf{C}\mid \bar{T}_1)\mathbf{C}_1$.

12.1.3 Let $\mathbf{K}$ denote the matrix obtained from the incidence matrix $\mathbf{M}$ of a connected digraph D by deleting any one of its rows. Show that $\mathbf{K}$ is a basis matrix of $\mathscr{B}$.

12.1.4 Show that if G is a plane graph, then $\mathscr{B}(G)\cong\mathscr{C}(G^*)$ and $\mathscr{C}(G)\cong\mathscr{B}(G^*)$.

12.1.5 A *circulation* of D *over a field* F is a function $f: A\rightarrow F$ which satisfies (12.1) in F; a *potential difference* of D *over* F is similarly defined. The vector spaces of these potential differences and circulations are denoted by $\mathscr{B}_F$ and $\mathscr{C}_F$. Show that theorem 12.2 remains valid if $\mathscr{B}$ and $\mathscr{C}$ are replaced by $\mathscr{B}_F$ and $\mathscr{C}_F$, respectively.

12.2 THE NUMBER OF SPANNING TREES

In this section we shall derive a formula for the number of spanning trees in a graph.

Let G be a connected graph and let T be a fixed spanning tree of G. Consider an arbitrary orientation D of G and let $\mathbf{B}$ be the basis matrix of $\mathscr{B}$ corresponding to T. It follows from theorem 12.2 that if S is a subset of A with $|S|=\nu-1$ then the square submatrix $\mathbf{B}\mid S$ is nonsingular if and only if S is a spanning tree of G. Thus the number of spanning trees of G is equal to the number of nonsingular submatrices of $\mathbf{B}$ of order $\nu-1$.

A matrix is said to be *unimodular* if all its full square submatrices have determinants 0, +1 or −1. The proof of the following theorem is due to Tutte (1965*b*).

Theorem 12.3 The basis matrix $\mathbf{B}$ is unimodular.

Proof Let $\mathbf{P}$ be a full submatrix of $\mathbf{B}$ (one of order $\nu-1$). Suppose that $\mathbf{P}=\mathbf{B}\mid T_1$. We may assume that T_1 is a spanning tree of D since, otherwise, $\det\mathbf{P}=0$ by theorem 12.2. Let $\mathbf{B}_1$ denote the basis matrix of $\mathscr{B}$ corresponding to T_1. Then (exercise 12.1.2*b*)

$$(\mathbf{B}\mid T_1)\mathbf{B}_1=\mathbf{B}$$

Restricting both sides to T, we obtain

$$(\mathbf{B}\mid T_1)(\mathbf{B}_1\mid T)=\mathbf{B}\mid T$$

Noting that $\mathbf{B}\mid T$ is an identity matrix, and taking determinants, we get

$$\det(\mathbf{B}\mid T_1)\det(\mathbf{B}_1\mid T)=1 \tag{12.5}$$

Both determinants in (12.5), being determinants of integer matrices, are themselves integers. It follows that $\det(\mathbf{B} \mid T_1) = \pm 1$ □

Theorem 12.4 $\tau(G) = \det \mathbf{BB}'$ (12.6)

Proof Using the formula for the determinant of the product of two rectangular matrices (see Hadley, 1961), we obtain

$$\det \mathbf{BB}' = \sum_{\substack{S \subseteq A \\ |S| = \nu - 1}} (\det(\mathbf{B} \mid S))^2 \tag{12.7}$$

Now, by theorem 12.2, the number of nonzero terms in (12.7) is equal to $\tau(G)$. But, by theorem 12.3, each such term has value 1 □

One can similarly show that if $\mathbf{C}$ is a basis matrix of $\mathscr{C}$ corresponding to a tree, then $\mathbf{C}$ is unimodular and

$$\tau(G) = \det \mathbf{CC}' \tag{12.8}$$

Corollary 12.4 $\tau(G) = \pm\det\begin{bmatrix} \mathbf{B} \\ \hdashline \mathbf{C} \end{bmatrix}$

Proof By (12.6) and (12.8)

$$(\tau(G))^2 = \det \mathbf{BB}' \det \mathbf{CC}' = \det\left[\begin{array}{c:c} \mathbf{BB}' & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{CC}' \end{array}\right]$$

Since $\mathscr{B}$ and $\mathscr{C}$ are orthogonal, $\mathbf{BC}' = \mathbf{CB}' = \mathbf{0}$. Thus

$$(\tau(G))^2 = \det\left[\begin{array}{c:c} \mathbf{BB}' & \mathbf{BC}' \\ \hdashline \mathbf{CB}' & \mathbf{CC}' \end{array}\right] = \det\left(\begin{bmatrix} \mathbf{B} \\ \hdashline \mathbf{C} \end{bmatrix}[\mathbf{B}' \mid \mathbf{C}']\right)$$

$$= \det\begin{bmatrix} \mathbf{B} \\ \hdashline \mathbf{C} \end{bmatrix} \det[\mathbf{B}' \mid \mathbf{C}'] = \left(\det\begin{bmatrix} \mathbf{B} \\ \hdashline \mathbf{C} \end{bmatrix}\right)^2$$

The corollary follows on taking square roots □

Since theorem 12.2 is valid for all basis matrices of $\mathscr{B}$, (12.6) clearly holds for any such matrix $\mathbf{B}$ that is unimodular. In particular, a matrix $\mathbf{K}$ obtained by deleting any one row of the incidence matrix $\boldsymbol{M}$ is unimodular (exercise 12.2.1*a*). Thus

$$\tau(G) = \det \mathbf{KK}'$$

This expression for the number of spanning trees in a graph is implicit in the work of Kirchhoff (1847), and is known as the *matrix-tree theorem*.

Exercises

12.2.1 Show that

(*a*)* a matrix **K** obtained from $\mathbf{M}$ by deleting any one row is unimodular;

(*b*) $\tau(G) = \pm\det\begin{bmatrix}\mathbf{K}\\ \text{---}\\ \mathbf{C}\end{bmatrix}$

12.2.2 The *conductance matrix* $\mathbf{C} = [c_{ij}]$ of a loopless graph G is the $\nu\times\nu$ matrix in which

$$c_{ii} = \sum_{j\neq i} a_{ij} \quad \text{for all} \quad i$$

$$c_{ij} = -a_{ij} \quad \text{for all} \quad i \quad \text{and} \quad j \quad \text{with} \quad i\neq j$$

where $\mathbf{A} = [a_{ij}]$ is the adjacency matrix of G. Show that

(*a*) $\mathbf{C} = \mathbf{M}\mathbf{M}'$, where $\mathbf{M}$ is the incidence matrix of any orientation of G;

(*b*) all cofactors of $\mathbf{C}$ are equal to $\tau(G)$.

12.2.3 A matrix is *totally unimodular* if all square submatrices have determinants 0, +1 or −1. Show that

(*a*) any basis matrix of $\mathscr{B}$ or $\mathscr{C}$ corresponding to a tree is totally unimodular;

(*b*) the incidence matrix of a simple graph G is totally unimodular if and only if G is bipartite.

12.2.4 Let F be a field of characteristic p. Show that

(*a*) if **B** and **C** are basis matrices of $\mathscr{B}_F$ and $\mathscr{C}_F$, respectively, corresponding to a tree, then $\det\begin{bmatrix}\mathbf{B}\\ \text{---}\\ \mathbf{C}\end{bmatrix} = \pm\tau(G)(\text{mod } p)$;

(*b*) $\dim(\mathscr{B}_F \cap \mathscr{C}_F) > 0$ if and only if $p \mid \tau(G)$. (H. Shank)

APPLICATIONS

12.3 PERFECT SQUARES

A *squared rectangle* is a rectangle dissected into at least two (but a finite number of) squares. If no two of the squares in the dissection have the same size, then the squared rectangle is *perfect*. The *order* of a squared rectangle is the number of squares into which it is dissected. Figure 12.7 shows a perfect rectangle of order 9. A squared rectangle is *simple* if it does not contain a rectangle which is itself squared. Clearly, every squared rectangle is composed of ones that are simple.

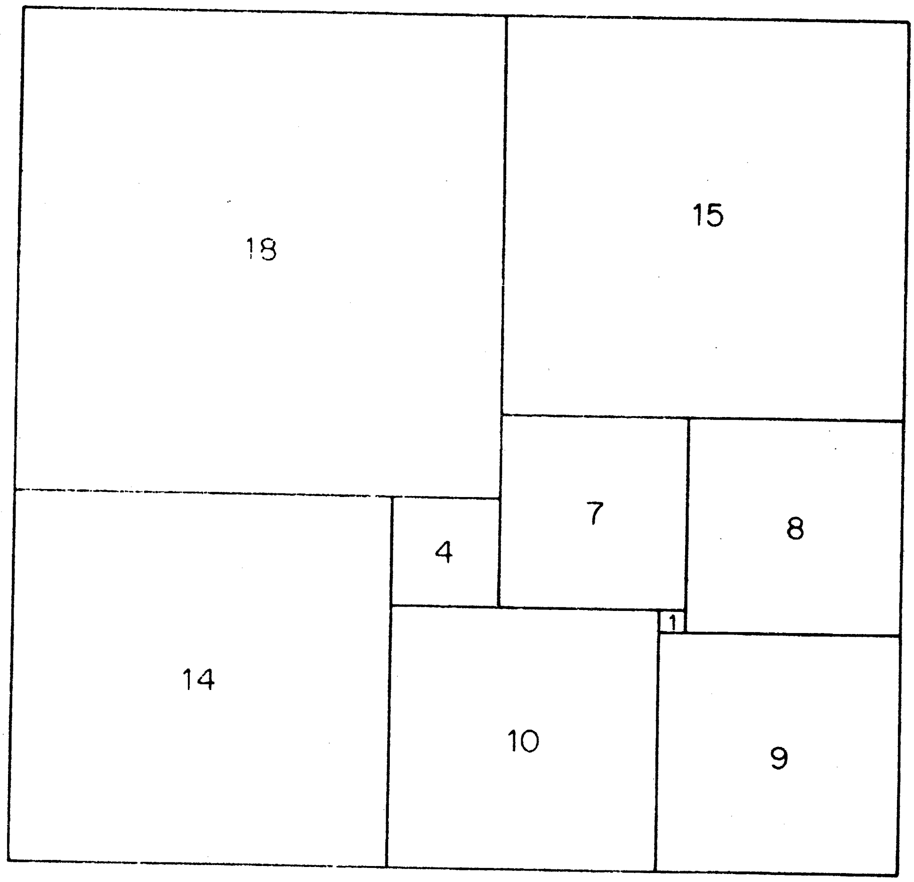


Figure 12.7. A perfect rectangle

For a long time no perfect squares were known, and it was conjectured that such squares did not exist. Sprague (1939) was the first to publish an example of a perfect square. About the same time, Brooks et al. (1940) developed systematic methods for their construction by using the theory of graphs. In this section, we shall present a brief discussion of their methods.

We first show how a digraph can be associated with a given squared rectangle R. The union of the horizontal sides of the constituent squares in the dissection consists of horizontal line segments; each such segment is called a *horizontal dissector* of R. In figure 12.8a, the horizontal dissectors are indicated by solid lines. We can now define the digraph D associated with R. To each horizontal dissector of R there corresponds a vertex of D; two vertices v_i and v_j of D are joined by an arc (v_i, v_j) if and only if their corresponding horizontal dissectors H_i and H_j flank some square of the dissection and H_i lies above H_j in R. Figure 12.8b shows the digraph associated with the squared rectangle in figure 12.8a. The vertices corresponding to the upper and lower sides of R are called the *poles* of D and are denoted by x and y, respectively.

We now assign to each vertex v of D a potential $p(v)$ equal to the height (above the lower side of R) of the corresponding horizontal dissector. If we regard D as an electrical network in which each wire has unit resistance, the potential difference $g = \delta p$ determines a flow of currents from x to y (see

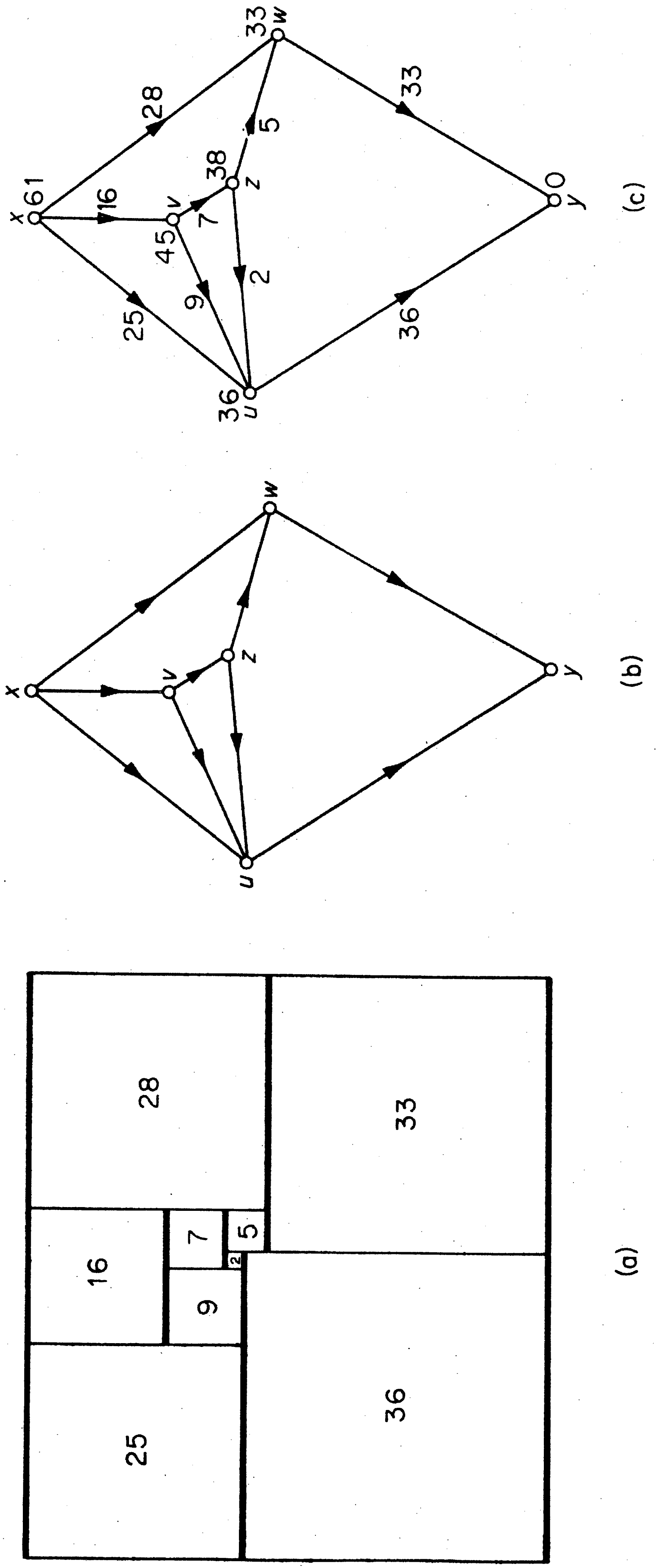

28
16
7
5
2
9
25
33
36
(a)
x
v
z
w
u
y
(b)
x 61
16
v 45
7
38 z
5
33 w
28
33
25
9
2
36 u
36
y 0
(c)

Figure 12.8

figure 12.8c). These currents satisfy *Kirchhoff's current law:* the total amount of current entering a vertex $v \in V\backslash\{x, y\}$ is equal to the total amount leaving it. For example, the total amount entering u in figure 12.8c is $25+9+2=36$, and the same amount leaves this vertex.

Let D be the digraph corresponding to a squared rectangle R, with poles x and y, and let G be the underlying graph of D. Then the graph $G+xy$ is called the *horizontal graph* of R. Brooks et al. (1940) showed that the horizontal graph of any simple squared rectangle is a 3-connected planar graph (their definition of connectivity differs slightly from the one used in this book). They also showed that, conversely, if H is a 3-connected planar graph and $xy \in E(H)$, then any flow of currents from x to y in $H-xy$ determines a squared rectangle. Thus one possible way of searching for perfect rectangles of order n is to

(i) list all 3-connected planar graphs with $n+1$ edges, and
(ii) for each such graph H and each edge xy of H, determine a flow of currents from x to y in $H-xy$.

Tutte (1961) showed that every 3-connected planar graph can be derived from a wheel by a sequence of operations involving face subdivisions and the taking of duals. Bouwkamp, Duijvestijn and Medema (1960) then applied Tutte's theorem to list all 3-connected planar graphs with at most 16 edges. Here we shall see how the theory developed in sections 12.1 and 12.2 can be used in computing a flow of currents from x to y in a digraph D.

Let $g(a)$ denote the current in arc a of D, and suppose that the total current leaving x is σ. Then

$$\sum_{a \in A} m_x(a)g(a) = \sigma \tag{12.9}$$

Kirchhoff's current law can be formulated as

$$\sum_{a \in A} m_v(a)g(a) = 0 \quad \text{for all} \quad v \in V\backslash\{x, y\} \tag{12.10}$$

Now, since g is a potential difference, it is orthogonal to every circulation. Therefore,

$$\mathbf{C}\mathrm{g}' = \mathbf{0} \tag{12.11}$$

where $\mathbf{C}$ is a basis matrix of $\mathscr{C}$ corresponding to a tree T of D and g$'$ is the transpose of the vector g. Equations (12.9)–(12.11) together give the matrix equation

$$\begin{bmatrix} \mathbf{K} \\ --- \\ \mathbf{C} \end{bmatrix} \mathrm{g}' = \begin{bmatrix} \sigma \\ --- \\ \mathbf{0} \end{bmatrix} \tag{12.12}$$

where $\mathbf{K}$ is the matrix obtained from $\boldsymbol{M}$ by deleting the row m_y. This

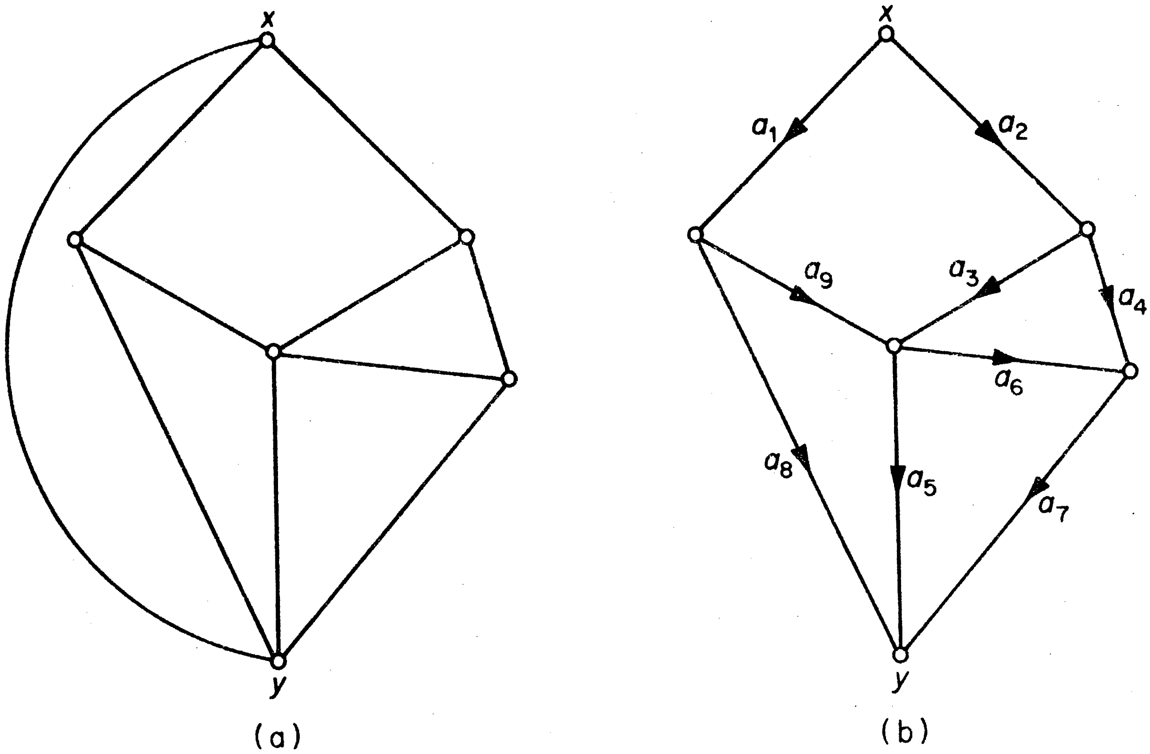


Figure 12.9

equation can be solved using Cramér's rule. Note that, since $\det\begin{bmatrix}\mathbf{K}\\ \text{---}\\ \mathbf{C}\end{bmatrix} = \pm\tau(G)$ (exercise 12.2.1*b*), we obtain a solution in integers if $\sigma = \tau(G)$. Thus, in computing the currents, it is convenient to take the total current leaving x to be equal to the number of spanning trees of D.

We illustrate the above procedure with an example. Consider the 3-connected planar graph in figure 12.9*a*. On deleting the edge xy and orienting each edge we obtain the digraph D of figure 12.9*b*.

It can be checked that the number of spanning trees in D is 66. By considering the tree $T = \{a_1, a_2, a_3, a_4, a_5\}$ we obtain the following nine equations, as in (12.12), (with $g(a_i)$ written simply as g_i).

$$
\begin{array}{l}
g_1 + g_2 = 66\\
g_1 - g_8 - g_9 = 0\\
g_2 - g_3 - g_4 = 0\\
g_3 - g_5 - g_6 + g_9 = 0\\
g_4 + g_6 - g_7 = 0\\
g_3 - g_4 + g_6 = 0\\
-g_3 + g_4 - g_5 + g_7 = 0\\
g_1 - g_2 - g_3 - g_5 + g_8 = 0\\
g_1 - g_2 - g_3 + g_9 = 0
\end{array}
$$

The solution to this system of equations is given by

$$(g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9) = (36, 30, 14, 16, 20, 2, 18, 28, 8)$$

The squared rectangle based on this flow of currents is just the one in figure 12.7 with all dimensions doubled.

Figure 12.10 shows a simple perfect square of order 25. It was discovered by Wilson (1967), and is the smallest (least order) such square known.

Further results on perfect squares can be found in the survey article by Tutte (1965a).

Exercises

12.3.1 Show that the constituent squares in a squared rectangle have commensurable sides.

12.3.2 The *vertical graph* of a squared rectangle R is the horizontal graph of the squared rectangle obtained by rotating R through a right angle. If no point of R is the corner of four constituent squares, show that the horizontal and vertical graphs of R are duals.

12.3.3* A *perfect cube* is a cube dissected into a finite number of smaller cubes, no two of the same size. Show that there exists no perfect cube.

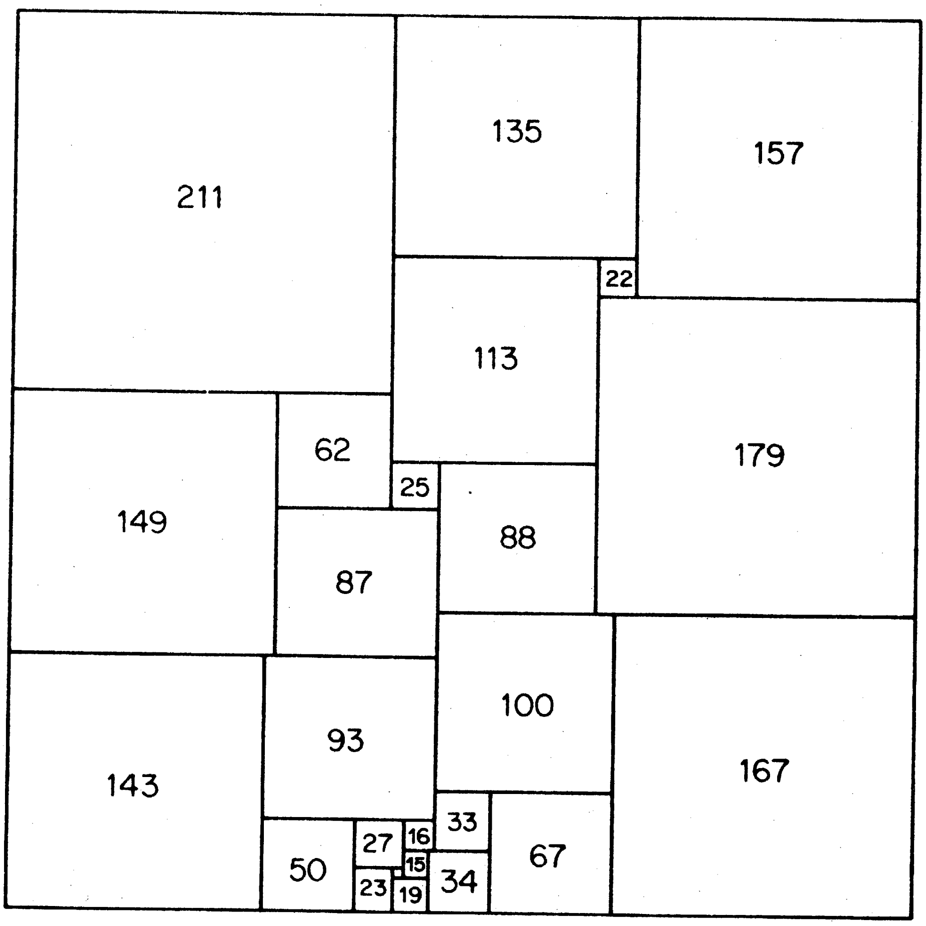


Figure 12.10. A simple perfect square of order 25

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