5 Matchings

5.1 MATCHINGS

A subset M of E is called a *matching* in G if its elements are links and no two are adjacent in G; the two ends of an edge in M are said to be *matched under* M. A matching M *saturates* a vertex v, and v is said to be M-*saturated*, if some edge of M is incident with v; otherwise, v is M-*unsaturated*. If every vertex of G is M-saturated, the matching M is *perfect*. M is a *maximum matching* if G has no matching M' with $|M'|>|M|$; clearly, every perfect matching is maximum. Maximum and perfect matchings in graphs are indicated in figure 5.1.

Let M be a matching in G. An M-*alternating path* in G is a path whose edges are alternately in $E\backslash M$ and M. For example, the path $v_5v_8v_1v_7v_6$ in the graph of figure 5.1*a* is an M-alternating path. An M-*augmenting path* is an M-alternating path whose origin and terminus are M-unsaturated.

Theorem 5.1 (Berge, 1957) A matching M in G is a maximum matching if and only if G contains no M-augmenting path.

Proof Let M be a matching in G, and suppose that G contains an M-augmenting path $v_0v_1 \ldots v_{2m+1}$. Define $M' \subseteq E$ by

$$M' = (M\backslash\{v_1v_2, v_3v_4, \ldots, v_{2m-1}v_{2m}\}) \cup \{v_0v_1, v_2v_3, \ldots, v_{2m}v_{2m+1}\}$$

Then M' is a matching in G, and $|M'| = |M|+1$. Thus M is not a maximum matching.

Conversely, suppose that M is not a maximum matching, and let M' be a maximum matching in G. Then

$$|M'|>|M| \tag{5.1}$$

Set $H = G[M \Delta M']$, where $M \Delta M'$ denotes the symmetric difference of M and M' (see figure 5.2).

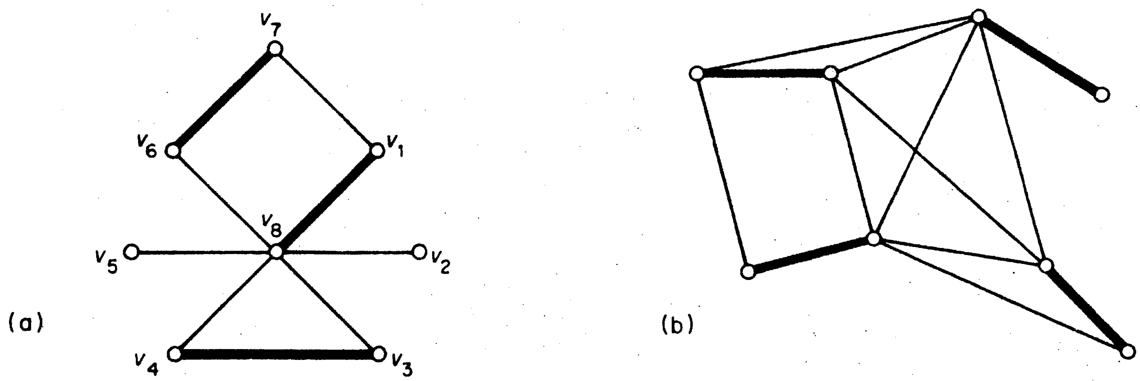


Figure 5.1. (*a*) A maximum matching; (*b*) a perfect matching

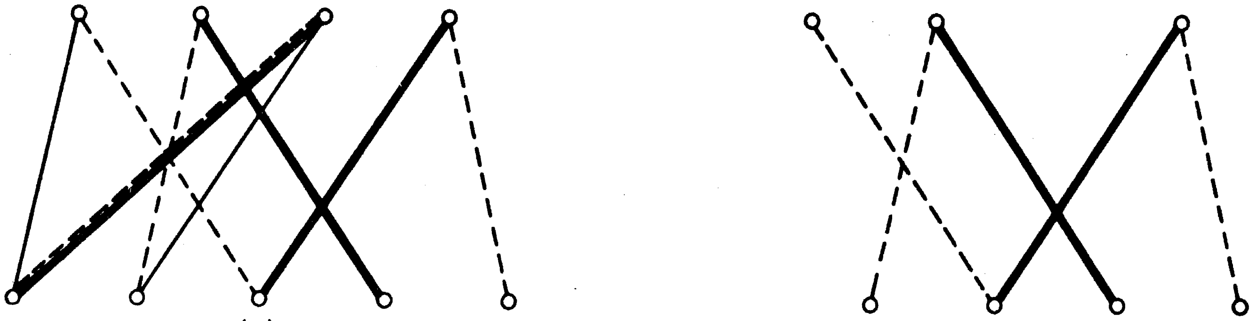


Figure 5.2. (*a*) G, with M heavy and M' broken; (*b*) $G[M \Delta M']$

Each vertex of H has degree either one or two in H, since it can be incident with at most one edge of M and one edge of M'. Thus each component of H is either an even cycle with edges alternately in M and M', or else a path with edges alternately in M and M'. By (5.1), H contains more edges of M' than of M, and therefore some path component P of H must start and end with edges of M'. The origin and terminus of P, being M'-saturated in H, are M-unsaturated in G. Thus P is an M-augmenting path in G □

Exercises

5.1.1 (*a*) Show that every k-cube has a perfect matching ($k \geq 2$).
(*b*) Find the number of different perfect matchings in K_{2n} and $K_{n,n}$.

5.1.2 Show that a tree has at most one perfect matching.

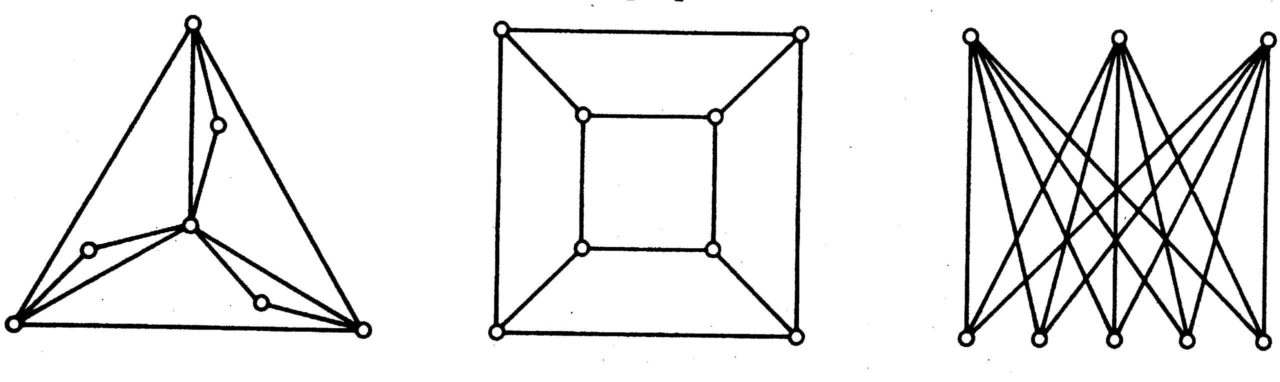
5.1.3 For each $k > 1$, find an example of a k-regular simple graph that has no perfect matching.

5.1.4 Two people play a game on a graph G by alternately selecting distinct vertices $v_0, v_1, v_2, \ldots$ such that, for $i > 0$, v_i is adjacent to v_{i-1}. The last player able to select a vertex wins. Show that the first player has a winning strategy if and only if G has no perfect matching.

5.1.5 A *k-factor* of G is a k-regular spanning subgraph of G, and G is *k-factorable* if there are edge-disjoint k-factors $H_1, H_2, \ldots, H_n$ such that $G = H_1 \cup H_2 \cup \ldots \cup H_n$.

(*a*)* Show that
(i) $K_{n,n}$ and K_{2n} are 1-factorable;
(ii) the Petersen graph is not 1-factorable.

(*b*) Which of the following graphs have 2-factors?



(c) Using Dirac's theorem (4.3), show that if G is simple, with ν even and $\delta \geq (\nu/2)+1$, then G has a 3-factor.

5.1.6* Show that K_{2n+1} can be expressed as the union of n connected 2-factors ($n \geq 1$).

5.2 MATCHINGS AND COVERINGS IN BIPARTITE GRAPHS

For any set S of vertices in G, we define the *neighbour set* of S in G to be the set of all vertices adjacent to vertices in S; this set is denoted by $N_G(S)$. Suppose, now, that G is a bipartite graph with bipartition (X, Y). In many applications one wishes to find a matching of G that saturates every vertex in X; an example is the personnel assignment problem, to be discussed in section 5.4. Necessary and sufficient conditions for the existence of such a matching were first given by Hall (1935).

Theorem 5.2 Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if

$$|N(S)| \geq |S| \quad \text{for all} \quad S \subseteq X \tag{5.2}$$

Proof Suppose that G contains a matching M which saturates every vertex in X, and let S be a subset of X. Since the vertices in S are matched under M with distinct vertices in $N(S)$, we clearly have $|N(S)| \geq |S|$.

Conversely, suppose that G is a bipartite graph satisfying (5.2), but that G contains no matching saturating all the vertices in X. We shall obtain a contradiction. Let M^* be a maximum matching in G. By our supposition, M^* does not saturate all vertices in X. Let u be an M^*-unsaturated vertex in X, and let Z denote the set of all vertices connected to u by M^*-alternating paths. Since M^* is a maximum matching, it follows from theorem 5.1 that u is the only M^*-unsaturated vertex in Z. Set $S = Z \cap X$ and $T = Z \cap Y$ (see figure 5.3).

Clearly, the vertices in $S \backslash \{u\}$ are matched under M^* with the vertices in T. Therefore

$$|T| = |S| - 1 \tag{5.3}$$

and $N(S) \supseteq T$. In fact, we have

$$N(S) = T \tag{5.4}$$

since every vertex in $N(S)$ is connected to u by an M^*-alternating path. But

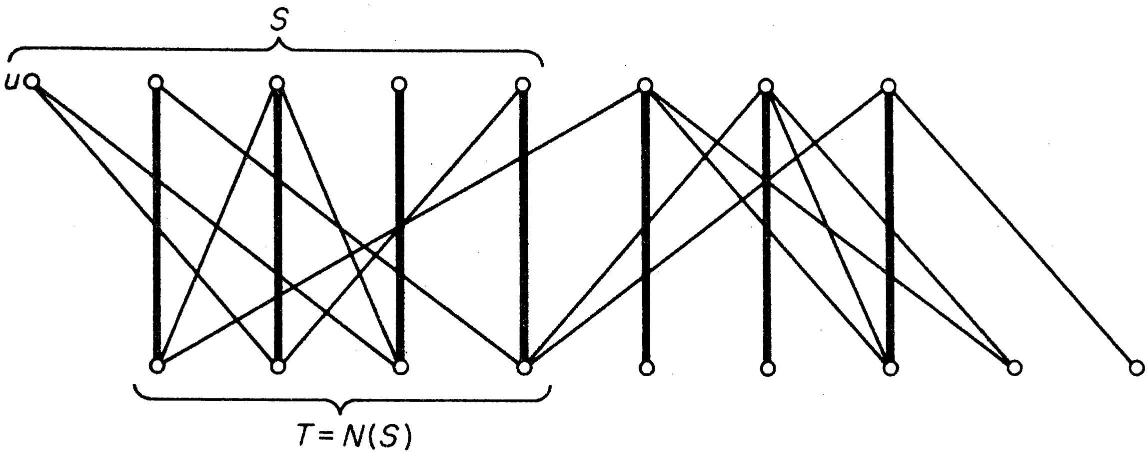


Figure 5.3

(5.3) and (5.4) imply that

$$|N(S)| = |S| - 1 < |S|$$

contradicting assumption (5.2) □

The above proof provides the basis of a good algorithm for finding a maximum matching in a bipartite graph. This algorithm will be presented in section 5.4.

Corollary 5.2 If G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.

Proof Let G be a k-regular bipartite graph with bipartition (X, Y). Since G is k-regular, $k\,|X| = |E| = k\,|Y|$ and so, since $k > 0$, $|X| = |Y|$. Now let S be a subset of X and denote by E_1 and E_2 the sets of edges incident with vertices in S and $N(S)$, respectively. By definition of $N(S)$, $E_1 \subseteq E_2$ and therefore

$$k\,|N(S)| = |E_2| \geq |E_1| = k\,|S|$$

It follows that $|N(S)| \geq |S|$ and hence, by theorem 5.2, that G has a matching M saturating every vertex in X. Since $|X| = |Y|$, M is a perfect matching □

Corollary 5.2 is sometimes known as the *marriage theorem*, since it can be more colourfully restated as follows: if every girl in a village knows exactly k boys, and every boy knows exactly k girls, then each girl can marry a boy she knows, and each boy can marry a girl he knows.

A *covering* of a graph G is a subset K of V such that every edge of G has at least one end in K. A covering K is a *minimum covering* if G has no covering K' with $|K'| < |K|$ (see figure 5.4).

If K is a covering of G, and M is a matching of G, then K contains at

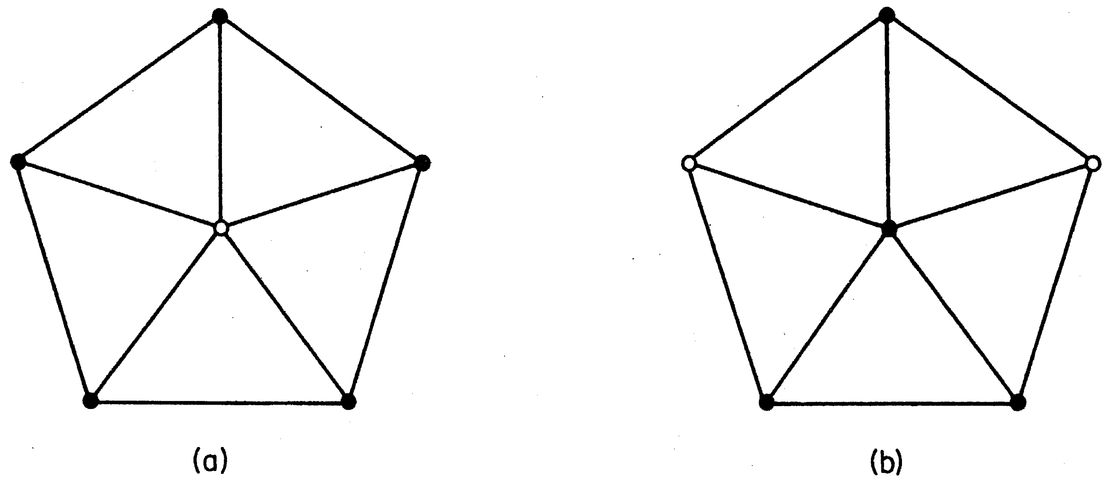


Figure 5.4. (*a*) A covering; (*b*) a minimum covering

least one end of each of the edges in M. Thus, for any matching M and any covering K, $|M| \le |K|$. Indeed, if M^* is a maximum matching and $\tilde{K}$ is a minimum covering, then

$$|M^*| \le |\tilde{K}| \tag{5.5}$$

In general, equality does not hold in (5.5) (see, for example, figure 5.4). However, if G is bipartite we do have $|M^*| = |\tilde{K}|$. This result, due to König (1931), is closely related to Hall's theorem. Before presenting its proof, we make a simple, but important, observation.

Lemma 5.3 Let M be a matching and K be a covering such that $|M| = |K|$. Then M is a maximum matching and K is a minimum covering.

Proof If M^* is a maximum matching and $\tilde{K}$ is a minimum covering then, by (5.5),

$$|M| \le |M^*| \le |\tilde{K}| \le |K|$$

Since $|M| = |K|$, it follows that $|M| = |M^*|$ and $|K| = |\tilde{K}|$ □

Theorem 5.3 In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.

Proof Let G be a bipartite graph with bipartition (X, Y), and let M^* be a maximum matching of G. Denote by U the set of M^*-unsaturated vertices in X, and by Z the set of all vertices connected by M^*-alternating paths to vertices of U. Set $S = Z \cap X$ and $T = Z \cap Y$. Then, as in the proof of theorem 5.2, we have that every vertex in T is M^*-saturated and $N(S) = T$. Define $\tilde{K} = (X \backslash S) \cup T$ (see figure 5.5). Every edge of G must have at least one of its ends in $\tilde{K}$. For, otherwise, there would be an edge with one end in

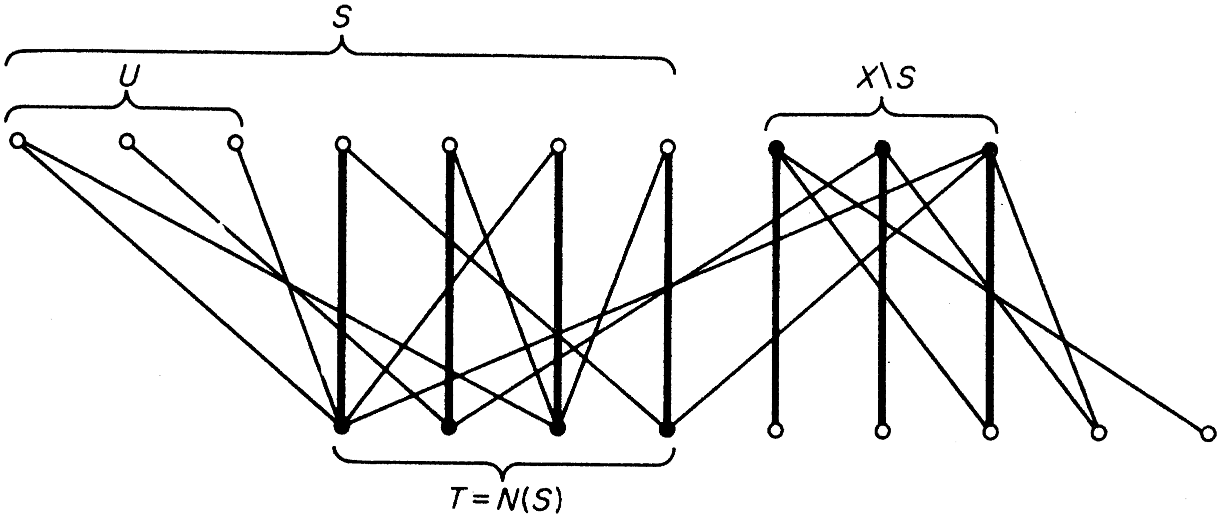


Figure 5.5

S and one end in $Y\backslash T$, contradicting $N(S)=T$. Thus $\tilde{K}$ is a covering of G and clearly

$$|M^*|=|\tilde{K}|$$

By lemma 5.3, $\tilde{K}$ is a minimum covering, and the theorem follows □

Exercises

5.2.1 Show that it is impossible, using 1×2 rectangles, to exactly cover an 8×8 square from which two opposite 1×1 corner squares have been removed.

5.2.2 (*a*) Show that a bipartite graph G has a perfect matching if and only if $|N(S)|\geq|S|$ for all $S\subseteq V$.

(*b*) Give an example to show that the above statement does not remain valid if the condition that G be bipartite is dropped.

5.2.3 For $k>0$, show that

(*a*) every k-regular bipartite graph is 1-factorable;

(*b*)* every $2k$-regular graph is 2-factorable. (J. Petersen)

5.2.4 Let $A_1, A_2, \ldots, A_m$ be subsets of a set S. A *system of distinct representatives* for the family $(A_1, A_2, \ldots, A_m)$ is a subset $\{a_1, a_2, \ldots, a_m\}$ of S such that $a_i\in A_i$, $1\leq i\leq m$, and $a_i\neq a_j$ for $i\neq j$. Show that $(A_1, A_2, \ldots, A_m)$ has a system of distinct representatives if and only if $\left|\bigcup_{i\in J} A_i\right|\geq|J|$ for all subsets J of $\{1, 2, \ldots, m\}$. (P. Hall)

5.2.5 A *line* of a matrix is a row or a column of the matrix. Show that the minimum number of lines containing all the 1's of a (0, 1)-matrix is equal to the maximum number of 1's, no two of which are in the same line.

5.2.6 (*a*) Prove the following generalisation of Hall's theorem (5.2): if G is a bipartite graph with bipartition (X, Y), the number of edges in a maximum matching of G is

$$|X| - \max_{S \subseteq X} \{|S| - |N(S)|\}$$

(D. König, O. Ore)

(*b*) Deduce that if G is simple with $|X| = |Y| = n$ and $\varepsilon > (k-1)n$, then G has a matching of cardinality k.

5.2.7 Deduce Hall's theorem (5.2) from König's theorem (5.3).

5.2.8* A non-negative real matrix $\mathbf{Q}$ is *doubly stochastic* if the sum of the entries in each row of $\mathbf{Q}$ is 1 and the sum of the entries in each column of $\mathbf{Q}$ is 1. A *permutation matrix* is a (0, 1)-matrix which has exactly one 1 in each row and each column. (Thus every permutation matrix is doubly stochastic.) Show that

(*a*) every doubly stochastic matrix is necessarily square;

(*b*) every doubly stochastic matrix $\mathbf{Q}$ can be expressed as a convex linear combination of permutation matrices; that is

$$\mathbf{Q} = c_1\mathbf{P}_1 + c_2\mathbf{P}_2 + \ldots + c_k\mathbf{P}_k$$

where each $\mathbf{P}_i$ is a permutation matrix, each c_i is a non-negative real number, and $\sum_1^k c_i = 1$. (G. Birkhoff, J. von Neumann)

5.2.9 Let H be a finite group and let K be a subgroup of H. Show that there exist elements $h_1, h_2, \ldots, h_n \in H$ such that $h_1K, h_2K, \ldots, h_nK$ are the left cosets of K and $Kh_1, Kh_2, \ldots, Kh_n$ are the right cosets of K. (P. Hall)

5.3 PERFECT MATCHINGS

A necessary and sufficient condition for a graph to have a perfect matching was obtained by Tutte (1947). The proof given here is due to Lovász (1973).

A component of a graph is *odd* or *even* according as it has an odd or even number of vertices. We denote by $o(G)$ the number of odd components of G.

Theorem 5.4 G has a perfect matching if and only if

$$o(G-S) \leq |S| \quad \text{for all} \quad S \subset V \tag{5.6}$$

Proof It clearly suffices to prove the theorem for simple graphs.

Suppose first that G has a perfect matching M. Let S be a proper subset of V, and let $G_1, G_2, \ldots, G_n$ be the odd components of $G-S$. Because G_i is odd, some vertex u_i of G_i must be matched under M with a vertex v_i of S (see figure 5.6). Therefore, since $\{v_1, v_2, \ldots, v_n\} \subseteq S$

$$o(G-S) = n = |\{v_1, v_2, \ldots, v_n\}| \leq |S|$$

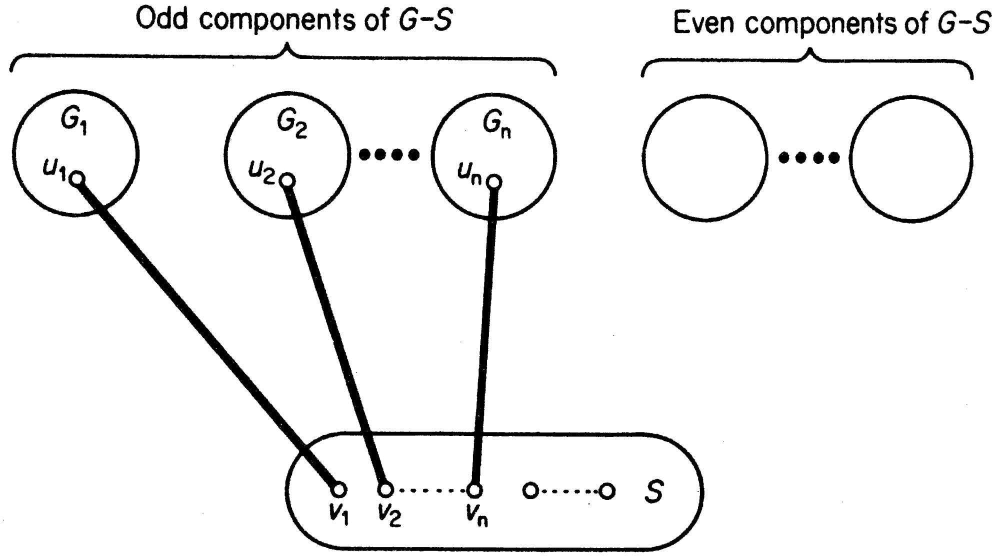


Figure 5.6

Conversely, suppose that G satisfies (5.6) but has no perfect matching. Then G is a spanning subgraph of a maximal graph G^* having no perfect matching. Since $G-S$ is a spanning subgraph of G^*-S we have $o(G^*-S) \leq o(G-S)$ and so, by (5.6),

$$o(G^*-S) \leq |S| \quad \text{for all} \quad S \subset V(G^*) \tag{5.7}$$

In particular, setting $S = \emptyset$, we see that $o(G^*) = 0$, and so $\nu(G^*)$ is even.

Denote by U the set of vertices of degree $\nu - 1$ in G^*. Since G^* clearly has a perfect matching if $U = V$, we may assume that $U \neq V$. We shall show that $G^* - U$ is a disjoint union of complete graphs. Suppose, to the contrary, that some component of $G^* - U$ is not complete. Then, in this component, there are vertices x, y and z such that $xy \in E(G^*)$, $yz \in E(G^*)$ and $xz \notin E(G^*)$ (exercise 1.6.14). Moreover, since $y \notin U$, there is a vertex w in $G^* - U$ such that $yw \notin E(G^*)$. The situation is illustrated in figure 5.7.

Since G^* is a maximal graph containing no perfect matching, $G^* + e$ has a perfect matching for all $e \notin E(G^*)$. Let M_1 and M_2 be perfect matchings in $G^* + xz$ and $G^* + yw$, respectively, and denote by H the subgraph of

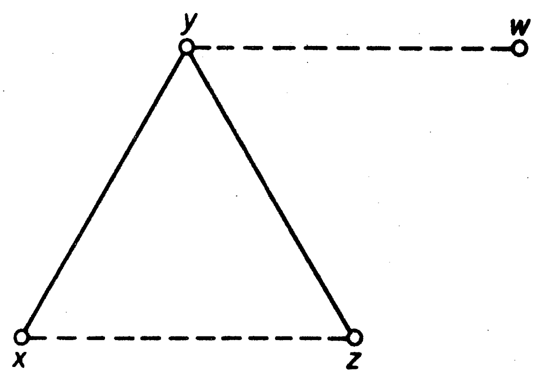


Figure 5.7

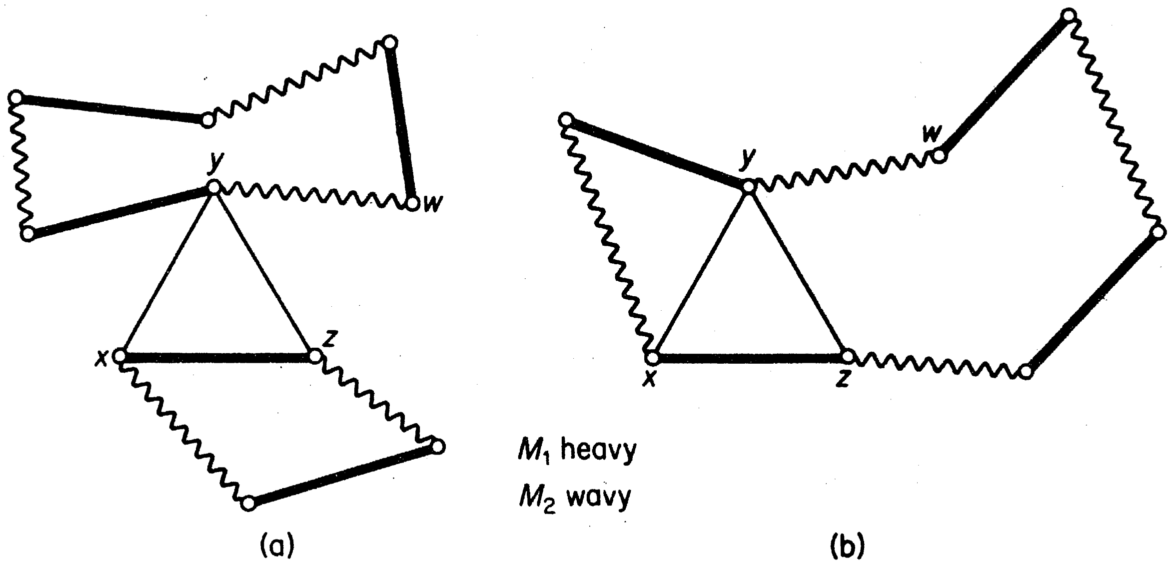


Figure 5.8

$G^* \cup \{xz, yw\}$ induced by $M_1 \Delta M_2$. Since each vertex of H has degree two, H is a disjoint union of cycles. Furthermore, all of these cycles are even, since edges of M_1 alternate with edges of M_2 around them. We distinguish two cases:

Case 1 xz and yw are in different components of H (figure 5.8a). Then, if yw is in the cycle C of H, the edges of M_1 in C, together with the edges of M_2 not in C, constitute a perfect matching in G^*, contradicting the definition of G^*.

Case 2 xz and yw are in the same component C of H. By symmetry of x and z, we may assume that the vertices x, y, w and z occur in that order on C (figure 5.8b). Then the edges of M_1 in the section $yw \ldots z$ of C, together with the edge yz and the edges of M_2 not in the section $yw \ldots z$ of C,

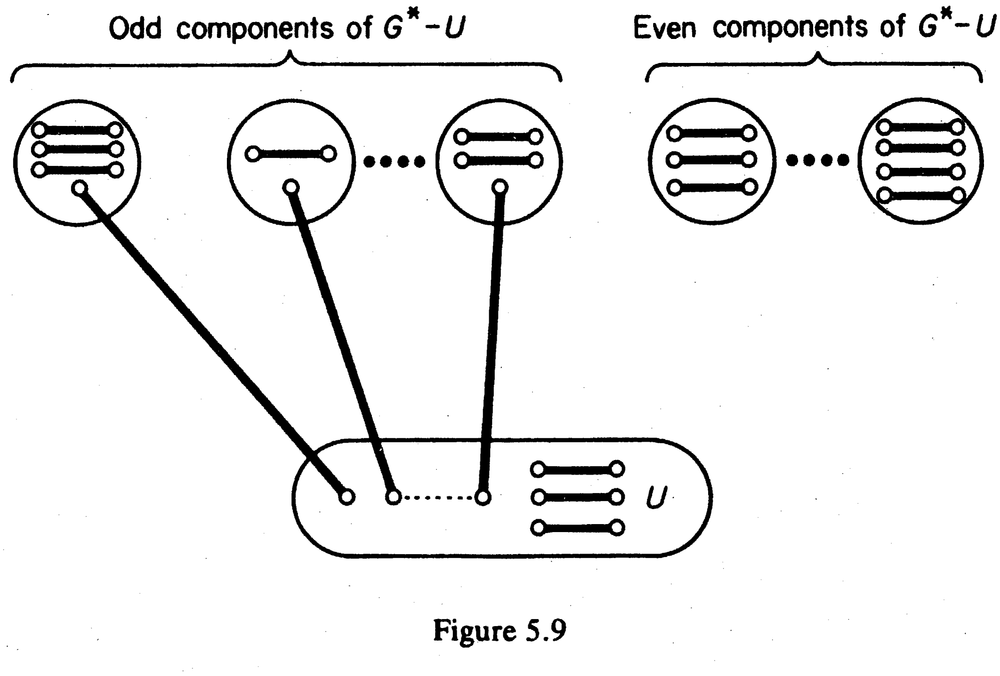


Figure 5.9

constitute a perfect matching in G^*, again contradicting the definition of G^*.

Since both case 1 and case 2 lead to contradictions, it follows that G^*-U is indeed a disjoint union of complete graphs.

Now, by (5.7), $o(G^*-U)\leq|U|$. Thus at most $|U|$ of the components of G^*-U are odd. But then G^* clearly has a perfect matching: one vertex in each odd component of G^*-U is matched with a vertex of U; the remaining vertices in U, and in components of G^*-U, are then matched as indicated in figure 5.9.

Since G^* was assumed to have no perfect matching we have obtained the desired contradiction. Thus G does indeed have a perfect matching □

The above theorem can also be proved with the aid of Hall's theorem (see Anderson, 1971).

From Tutte's theorem, we now deduce a result first obtained by Petersen (1891).

Corollary 5.4 Every 3-regular graph without cut edges has a perfect matching.

Proof Let G be a 3-regular graph without cut edges, and let S be a proper subset of V. Denote by $G_1, G_2, \ldots, G_n$ the odd components of $G-S$, and let m_i be the number of edges with one end in G_i and one end in S, $1\leq i\leq n$. Since G is 3-regular

$$\sum_{v\in V(G_i)} d(v)=3\nu(G_i) \quad \text{for} \quad 1\leq i\leq n \tag{5.8}$$

and

$$\sum_{v\in S} d(v)=3\,|S| \tag{5.9}$$

By (5.8), $m_i=\sum_{v\in V(G_i)} d(v)-2\varepsilon(G_i)$ is odd. Now $m_i\neq 1$ since G has no cut edge. Thus

$$m_i\geq 3 \quad \text{for} \quad 1\leq i\leq n \tag{5.10}$$

It follows from (5.10) and (5.9) that

$$o(G-S)=n\leq\frac{1}{3}\sum_{i=1}^{n} m_i\leq\frac{1}{3}\sum_{v\in S} d(v)=|S|$$

Therefore, by theorem 5.4, G has a perfect matching □

A 3-regular graph with cut edges need not have a perfect matching. For example, it follows from theorem 5.4 that the graph G of figure 5.10 has no perfect matching, since $o(G-v)=3$.

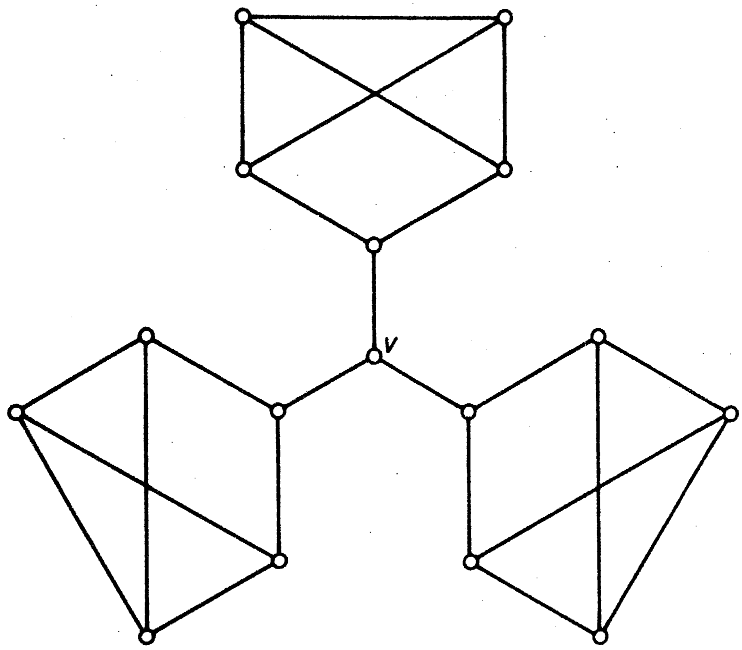


Figure 5.10

Exercises

5.3.1* Derive Hall's theorem (5.2) from Tutte's theorem (5.4).

5.3.2 Prove the following generalisation of corollary 5.4: if G is a $(k-1)$-edge-connected k-regular graph with ν even, then G has a perfect matching.

5.3.3 Show that a tree G has a perfect matching if and only if $o(G-v)=1$ for all $v \in V$. (V. Chungphaisan)

5.3.4* Prove the following generalisation of Tutte's theorem (5.4): the number of edges in a maximum matching of G is $\frac{1}{2}(\nu-d)$, where $d=\max\limits_{S\subset V}\{o(G-S)-|S|\}$. (C. Berge)

5.3.5 (*a*) Using Tutte's theorem (5.4), characterise the maximal simple graphs which have no perfect matching.

(*b*) Let G be simple, with ν even and $\delta<\nu/2$. Show that if $\varepsilon > \binom{\delta}{2}+\binom{\nu-2\delta-1}{2}+\delta(\nu-\delta)$, then G has a perfect matching.

APPLICATIONS

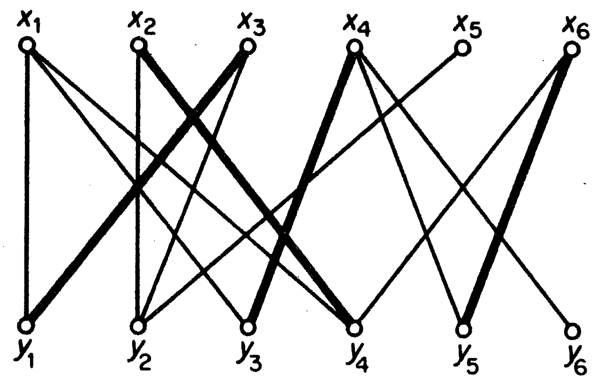
5.4 THE PERSONNEL ASSIGNMENT PROBLEM

In a certain company, n workers $X_1, X_2, \ldots, X_n$ are available for n jobs $Y_1, Y_2, \ldots, Y_n$, each worker being qualified for one or more of these jobs. Can all the men be assigned, one man per job, to jobs for which they are qualified? This is the *personnel assignment problem*.

We construct a bipartite graph G with bipartition (X, Y), where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, and x_i is joined to y_j if and only if worker X_i is qualified for job Y_j. The problem becomes one of determining whether or not G has a perfect matching. According to Hall's theorem (5.2), either G has such a matching or there is a subset S of X such that $|N(S)| < |S|$. In the sequel, we shall present an algorithm to solve the personnel assignment problem. Given any bipartite graph G with bipartition (X, Y), the algorithm either finds a matching of G that saturates every vertex in X or, failing this, finds a subset S of X such that $|N(S)| < |S|$.

The basic idea behind the algorithm is very simple. We start with an arbitrary matching M. If M saturates every vertex in X, then it is a matching of the required type. If not, we choose an M-unsaturated vertex u in X and systematically search for an M-augmenting path with origin u. Our method of search, to be described in detail below, finds such a path P if one exists; in this case $\hat{M} = M \,\Delta E(P)$ is a larger matching than M, and hence saturates more vertices in X. We then repeat the procedure with $\hat{M}$ instead of M. If such a path does not exist, the set Z of all vertices which are connected to u by M-alternating paths is found. Then (as in the proof of theorem 5.2) $S = Z \cap X$ satisfies $|N(S)| < |S|$.

Let M be a matching in G, and let u be an M-unsaturated vertex in X. A tree $H \subseteq G$ is called an *M-alternating tree rooted at u* if (i) $u \in V(H)$, and (ii) for every vertex v of H, the unique (u, v)-path in H is an M-alternating path. An M-alternating tree in a graph is shown in figure 5.11.



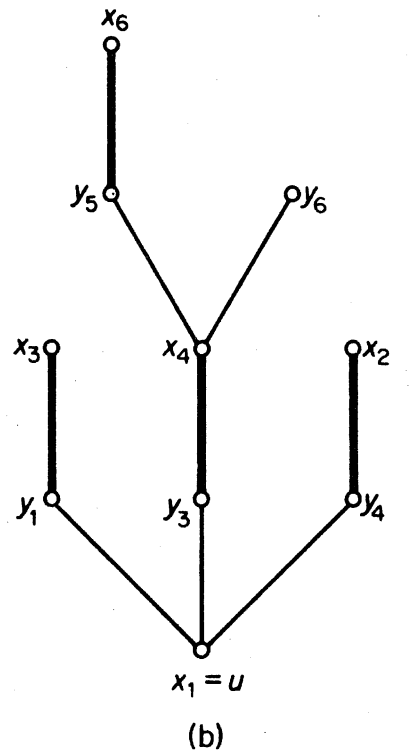


Figure 5.11. (*a*) A matching M in G; (*b*) an M-alternating tree in G

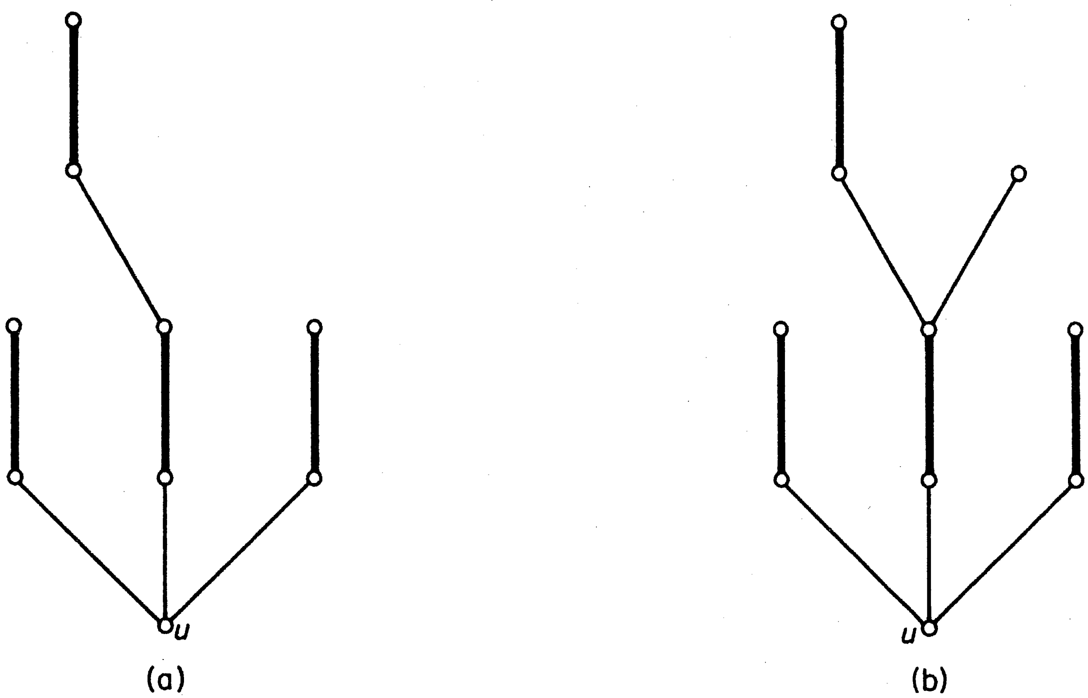


Figure 5.12. (*a*) Case (i); (*b*) case (ii)

The search for an M-augmenting path with origin u involves 'growing' an M-alternating tree H rooted at u. This procedure was first suggested by Edmonds (1965). Initially, H consists of just the single vertex u. It is then grown in such a way that, at any stage, either

(i) all vertices of H except u are M-saturated and matched under M (as in figure 5.12a), or
(ii) H contains an M-unsaturated vertex different from u (as in figure 5.12b).

If (i) is the case (as it is initially) then, setting $S = V(H) \cap X$ and $T = V(H) \cap Y$, we have $N(S) \supseteq T$; thus either $N(S) = T$ or $N(S) \supset T$.

(a) If $N(S) = T$ then, since the vertices in $S \backslash \{u\}$ are matched with the vertices in T, $|N(S)| = |S| - 1$, indicating that G has no matching saturating all vertices in X.
(b) If $N(S) \supset T$, there is a vertex y in $Y \backslash T$ adjacent to a vertex x in S. Since all vertices of H except u are matched under M, either $x = u$ or else x is matched with a vertex of H. Therefore $xy \notin M$. If y is M-saturated, with $yz \in M$, we grow H by adding the vertices y and z and the edges xy and yz. We are then back in case (i). If y is M-unsaturated, we grow H by adding the vertex y and the edge xy, resulting in case (ii). The (u, y)-path of H is then an M-augmenting path with origin u, as required.

Figure 5.13 illustrates the above tree-growing procedure.

The algorithm described above is known as the *Hungarian method*, and

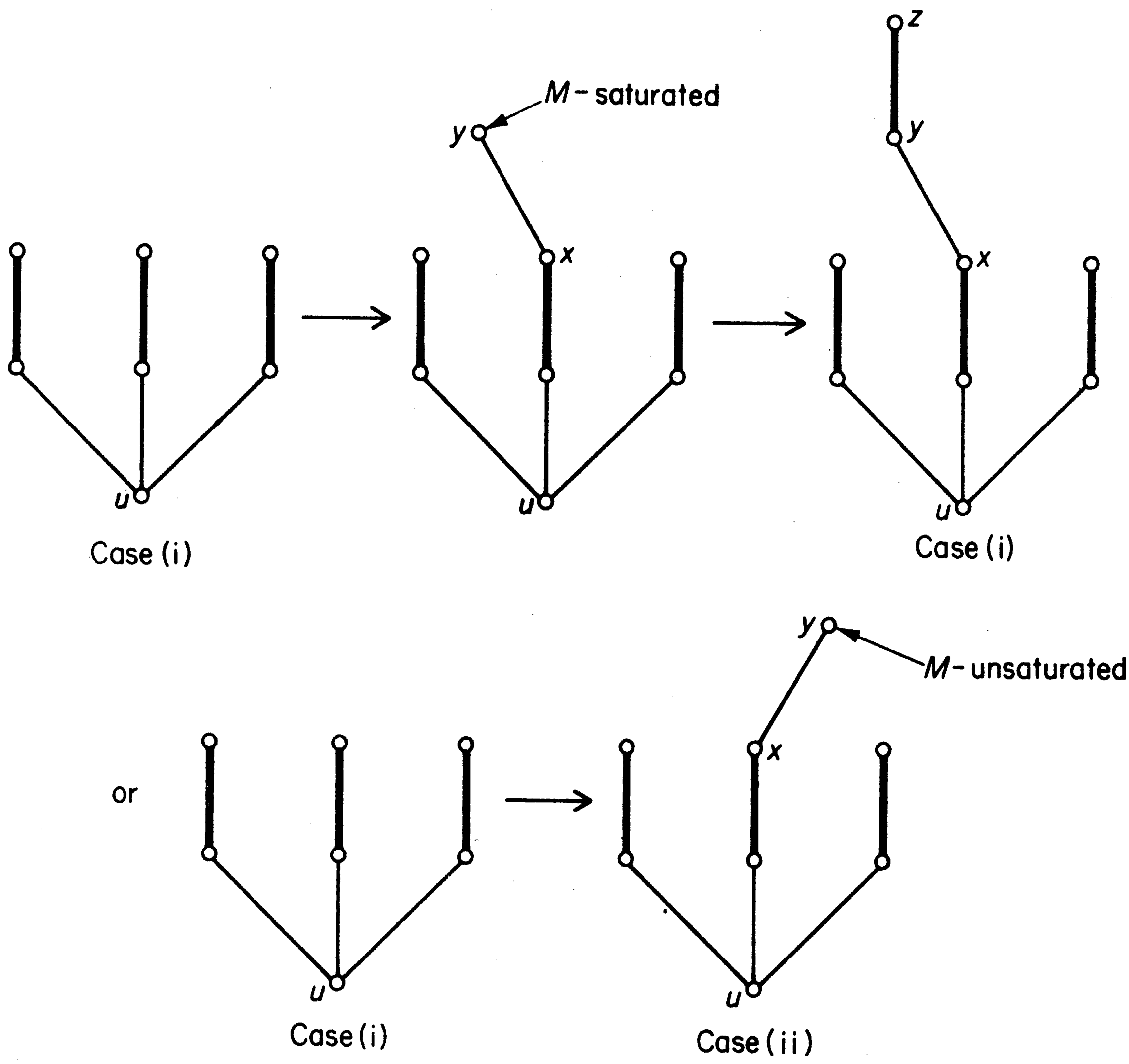


Figure 5.13. The tree-growing procedure

can be summarised as follows:

Start with an arbitrary matching M.

1. If M saturates every vertex in X, stop. Otherwise, let u be an M-unsaturated vertex in X. Set $S=\{u\}$ and $T=\emptyset$.
2. If $N(S)=T$ then $|N(S)|<|S|$, since $|T|=|S|-1$. Stop, since by Hall's theorem there is no matching that saturates every vertex in X. Otherwise, let $y \in N(S)\backslash T$.
3. If y is M-saturated, let $yz \in M$. Replace S by $S\cup\{z\}$ and T by $T\cup\{y\}$ and go to step 2. (Observe that $|T|=|S|-1$ is maintained after this replacement.) Otherwise, let P be an M-augmenting (u, y)-path. Replace M by $\hat{M}=M\,\Delta E(P)$ and go to step 1.

Consider, for example, the graph G in figure 5.14*a*, with initial matching $M=\{x_2y_2, x_3y_3, x_5y_5\}$. In figure 5.14*b* an M-alternating tree is grown, starting with x_1, and the M-augmenting path $x_1y_2x_2y_1$ found. This results in a new matching $\hat{M}=\{x_1y_2, x_2y_1, x_3y_3, x_5y_5\}$, and an $\hat{M}$-alternating tree is now grown from x_4 (figures 5.14*c* and 5.14*d*) Since there is no $\hat{M}$-augmenting

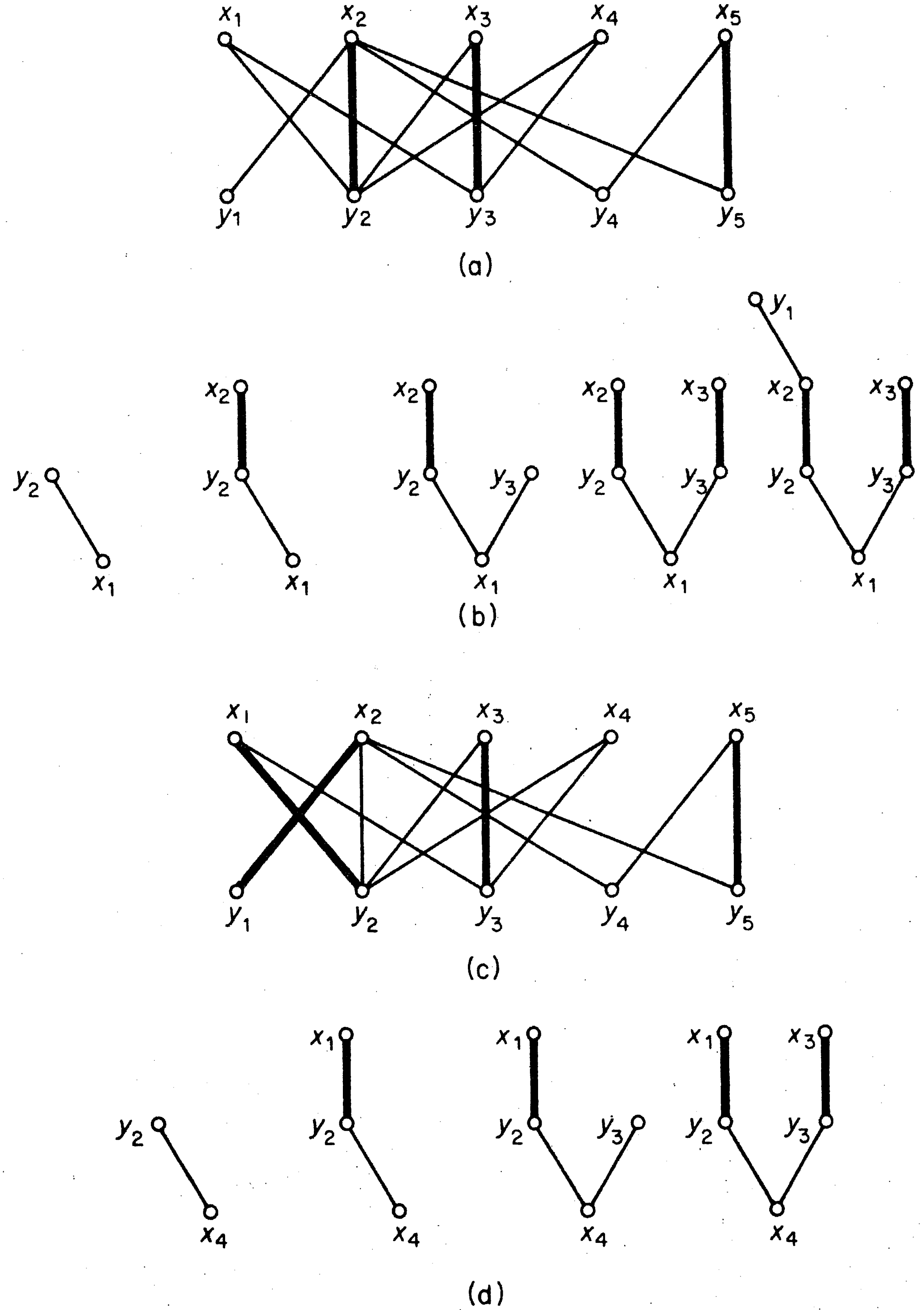


Figure 5.14. (*a*) Matching M; (*b*) an M-alternating tree; (*c*) matching $\hat{M}$; (*d*) an $\hat{M}$-alternating tree

path with origin x_4, the algorithm terminates. The set $S = \{x_1, x_3, x_4\}$, with neighbour set $N(S) = \{y_2, y_3\}$, shows that G has no perfect matching.

A flow diagram of the Hungarian method is given in figure 5.15. Since the algorithm can cycle through the tree-growing procedure, I, at most $|X|$ times before finding either an $S \subseteq X$ such that $|N(S)| < |S|$ or an M-augmenting path, and since the initial matching can be augmented at most $|X|$ times

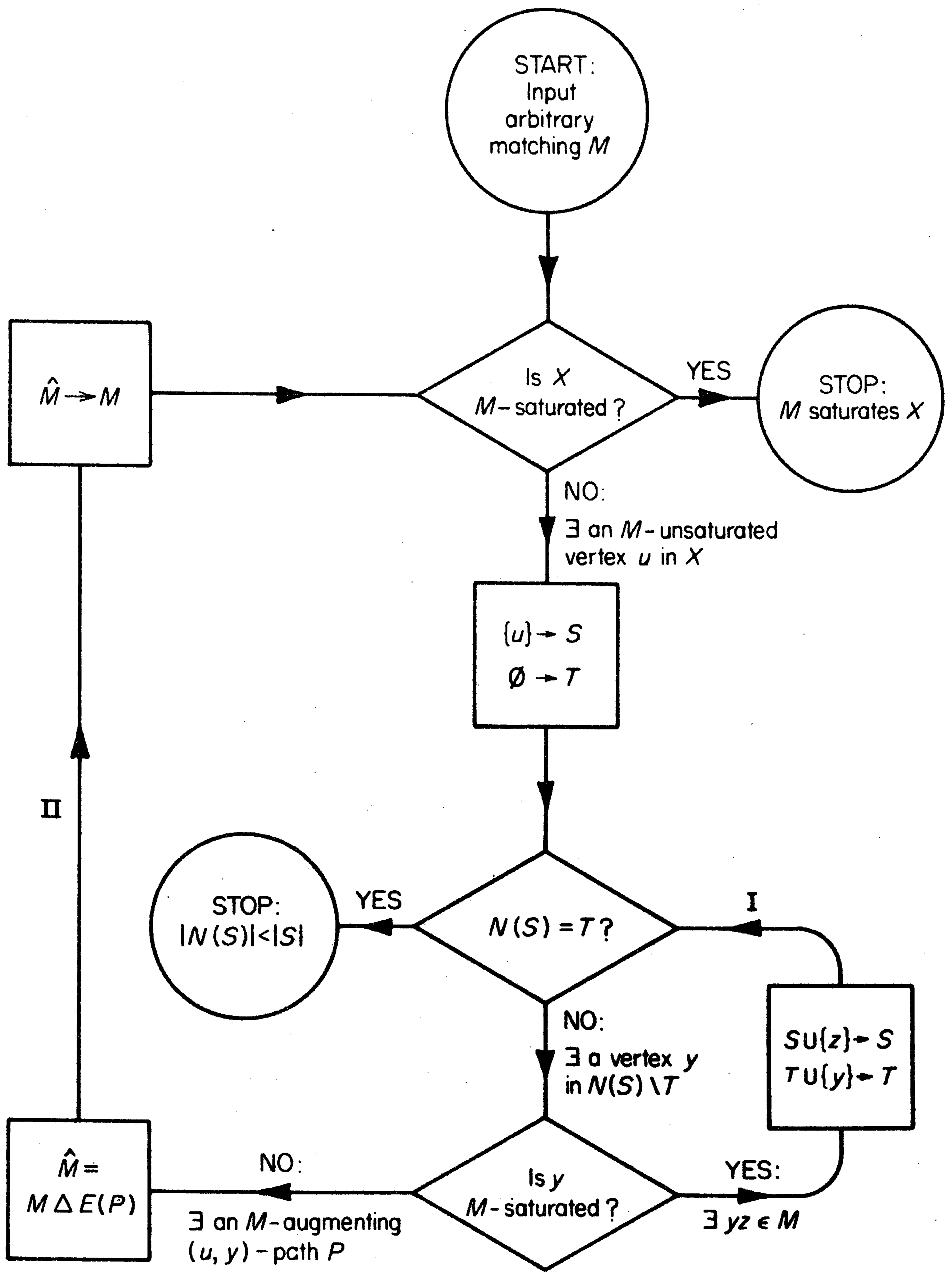


Figure 5.15. The Hungarian method

before a matching of the required type is found, it is clear that the Hungarian method is a good algorithm.

One can find a maximum matching in a bipartite graph by slightly modifying the above procedure (exercise 5.4.1). A good algorithm that determines such a matching in any graph has been given by Edmonds (1965).

Exercise

5.4.1 Describe how the Hungarian method can be used to find a maximum matching in a bipartite graph.

5.5 THE OPTIMAL ASSIGNMENT PROBLEM

The Hungarian method, described in section 5.4, is an efficient way of determining a feasible assignment of workers to jobs, if one exists. However one may, in addition, wish to take into account the effectiveness of the workers in their various jobs (measured, perhaps, by the profit to the company). In this case, one is interested in an assignment that maximises the total effectiveness of the workers. The problem of finding such an assignment is known as the *optimal assignment problem.*

Consider a weighted complete bipartite graph with bipartition (X, Y), where $X=\{x_1, x_2, \ldots, x_n\}$, $Y=\{y_1, y_2, \ldots, y_n\}$ and edge x_iy_j has weight $w_{ij}=w(x_iy_j)$, the effectiveness of worker X_i in job Y_j. The optimal assignment problem is clearly equivalent to that of finding a maximum-weight perfect matching in this weighted graph. We shall refer to such a matching as an *optimal matching.*

To solve the optimal assignment problem it is, of course, possible to enumerate all $n!$ perfect matchings and find an optimal one among them. However, for large n, such a procedure would clearly be most inefficient. In this section we shall present a good algorithm for finding an optimal matching in a weighted complete bipartite graph.

We define a *feasible vertex labelling* as a real-valued function l on the vertex set $X \cup Y$ such that, for all $x \in X$ and $y \in Y$

$$l(x)+l(y) \geq w(xy) \tag{5.11}$$

(The real number $l(v)$ is called the *label* of the vertex v.) A feasible vertex labelling is thus a labelling of the vertices such that the sum of the labels of the two ends of an edge is at least as large as the weight of the edge. No matter what the edge weights are, there always exists a feasible vertex labelling; one such is the function l given by

$$\left.\begin{aligned} l(x) &= \max_{y\in Y} w(xy) && \text{if} \quad x \in X \\ l(y) &= 0 && \text{if} \quad y \in Y \end{aligned}\right\} \tag{5.12}$$

If l is a feasible vertex labelling, we denote by E_l the set of those edges for which equality holds in (5.11); that is

$$E_l=\{xy \in E \mid l(x)+l(y)=w(xy)\}$$

The spanning subgraph of G with edge set E_l is referred to as the *equality subgraph* corresponding to the feasible vertex labelling l, and is denoted by G_l. The connection between equality subgraphs and optimal matchings is provided by the following theorem.

Theorem 5.5 Let l be a feasible vertex labelling of G. If G_l contains a perfect matching M^*, then M^* is an optimal matching of G.

Proof Suppose that G_l contains a perfect matching M^*. Since G_l is a spanning subgraph of G, M^* is also a perfect matching of G. Now

$$w(M^*) = \sum_{e\in M^*} w(e) = \sum_{v\in V} l(v) \tag{5.13}$$

since each $e \in M^*$ belongs to the equality subgraph and the ends of edges of M^* cover each vertex exactly once. On the other hand, if M is any perfect matching of G, then

$$w(M) = \sum_{e\in M} w(e) \leq \sum_{v\in V} l(v) \tag{5.14}$$

It follows from (5.13) and (5.14) that $w(M^*) \geq w(M)$. Thus M^* is an optimal matching □

The above theorem is the basis of an algorithm, due to Kuhn (1955) and Munkres (1957), for finding an optimal matching in a weighted complete bipartite graph. Our treatment closely follows Edmonds (1967).

Starting with an arbitrary feasible vertex labelling l (for example, the one given in (5.12)), we determine G_l, choose an arbitrary matching M in G_l and apply the Hungarian method. If a perfect matching is found in G_l then, by theorem 5.5, this matching is optimal. Otherwise, the Hungarian method terminates in a matching M' that is not perfect, and an M'-alternating tree H that contains no M'-augmenting path and cannot be grown further (in G_l). We then modify l to a feasible vertex labelling $\hat{l}$ with the property that both M' and H are contained in $G_{\hat{l}}$ and H can be extended in $G_{\hat{l}}$. Such modifications in the feasible vertex labelling are made whenever necessary, until a perfect matching is found in some equality subgraph.

The Kuhn–Munkres Algorithm

Start with an arbitrary feasible vertex labelling l, determine G_l, and choose an arbitrary matching M in G_l.

1. If X is M-saturated, then M is a perfect matching (since $|X| = |Y|$) and hence, by theorem 5.5, an optimal matching; in this case, stop. Otherwise, let u be an M-unsaturated vertex. Set $S = \{u\}$ and $T = \emptyset$.
2. If $N_{G_l}(S) \supset T$, go to step 3. Otherwise, $N_{G_l}(S) = T$. Compute

$$\alpha_l = \min_{\substack{x\in S\\ y\notin T}} \{l(x) + l(y) - w(xy)\}$$

and the feasible vertex labelling $\hat{l}$ given by

$$\hat{l}(v) = \begin{cases} l(v) - \alpha_l & \text{if} \quad v \in S \\ l(v) + \alpha_l & \text{if} \quad v \in T \\ l(v) & \text{otherwise} \end{cases}$$

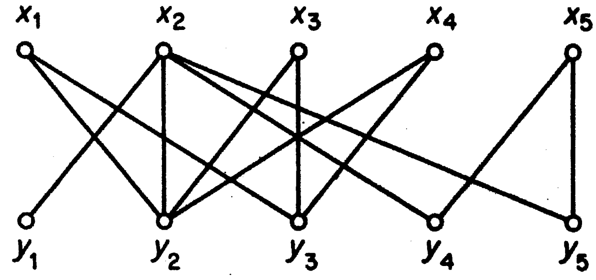
(Note that $\alpha_l > 0$ and that $N_{G_{\hat{l}}}(S) \supset T$.) Replace l by $\hat{l}$ and G_l by $G_{\hat{l}}$.

$$\begin{bmatrix} 3 & 5 & 5 & 4 & 1 \\ 2 & 2 & 0 & 2 & 2 \\ 2 & 4 & 4 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 3 & 3 \end{bmatrix}$$

(a)

$$\begin{array}{c c} \begin{bmatrix} 3 & \mathbf{5} & \mathbf{5} & 4 & 1 \\ \mathbf{2} & \mathbf{2} & 0 & \mathbf{2} & \mathbf{2} \\ 2 & \mathbf{4} & \mathbf{4} & 1 & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 1 & 2 & 1 & \mathbf{3} & \mathbf{3} \end{bmatrix} & \begin{array}{c} 5 \\ 2 \\ 4 \\ 1 \\ 3 \end{array} \\ \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array} & \end{array}$$

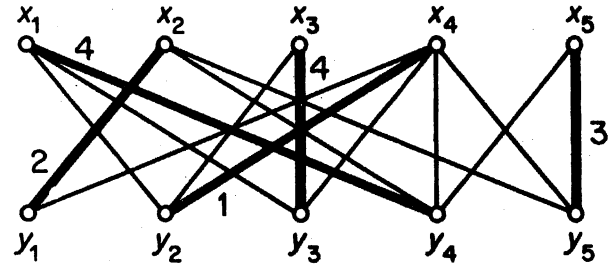
(b)



(c)

$$\begin{array}{c c} \begin{bmatrix} 3 & \mathbf{5} & \mathbf{5} & \mathbf{4} & 1 \\ \mathbf{2} & 2 & 0 & \mathbf{2} & \mathbf{2} \\ 2 & \mathbf{4} & \mathbf{4} & 1 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & 2 & 1 & \mathbf{3} & \mathbf{3} \end{bmatrix} & \begin{array}{c} 4 \\ 2 \\ 3 \\ 0 \\ 3 \end{array} \\ \begin{array}{ccccc} 0 & 1 & 1 & 0 & 0 \end{array} & \end{array}$$

(d)



(e)

Figure 5.16

3. Choose a vertex y in $N_{G_l}(S)\backslash T$. As in the tree-growing procedure of section 5.4, consider whether or not y is M-saturated. If y is M-saturated, with $yz \in M$, replace S by $S \cup \{z\}$ and T by $T \cup \{y\}$, and go to step 2. Otherwise, let P be an M-augmenting (u, y)-path in G_l, replace M by $\hat{M} = M \,\Delta E(P)$, and go to step 1.

In illustrating the Kuhn–Munkres algorithm, it is convenient to represent a weighted complete bipartite graph G by a matrix $\mathbf{W} = [w_{ij}]$, where w_{ij} is the weight of edge x_iy_j in G. We shall start with the matrix of figure 5.16*a*. In figure 5.16*b*, the feasible vertex labelling (5.12) is shown (by placing the label of x_i to the right of row i of the matrix and the label of y_j below column j) and the entries corresponding to edges of the associated equality subgraph are indicated; the equality subgraph itself is depicted (without weights) in figure 5.16*c*. It was shown in the previous section that this graph has no perfect matching (the set $S = \{x_1, x_3, x_4\}$ has neighbour set $\{y_2, y_3\}$). We therefore modify our initial feasible vertex labelling to the one given in figure 5.16*d*. An application of the Hungarian method now shows that the associated equality subgraph (figure 5.16*e*) has the perfect matching $\{x_1y_4, x_2y_1, x_3y_3, x_4y_2, x_5y_5\}$. This is therefore an optimal matching of G.

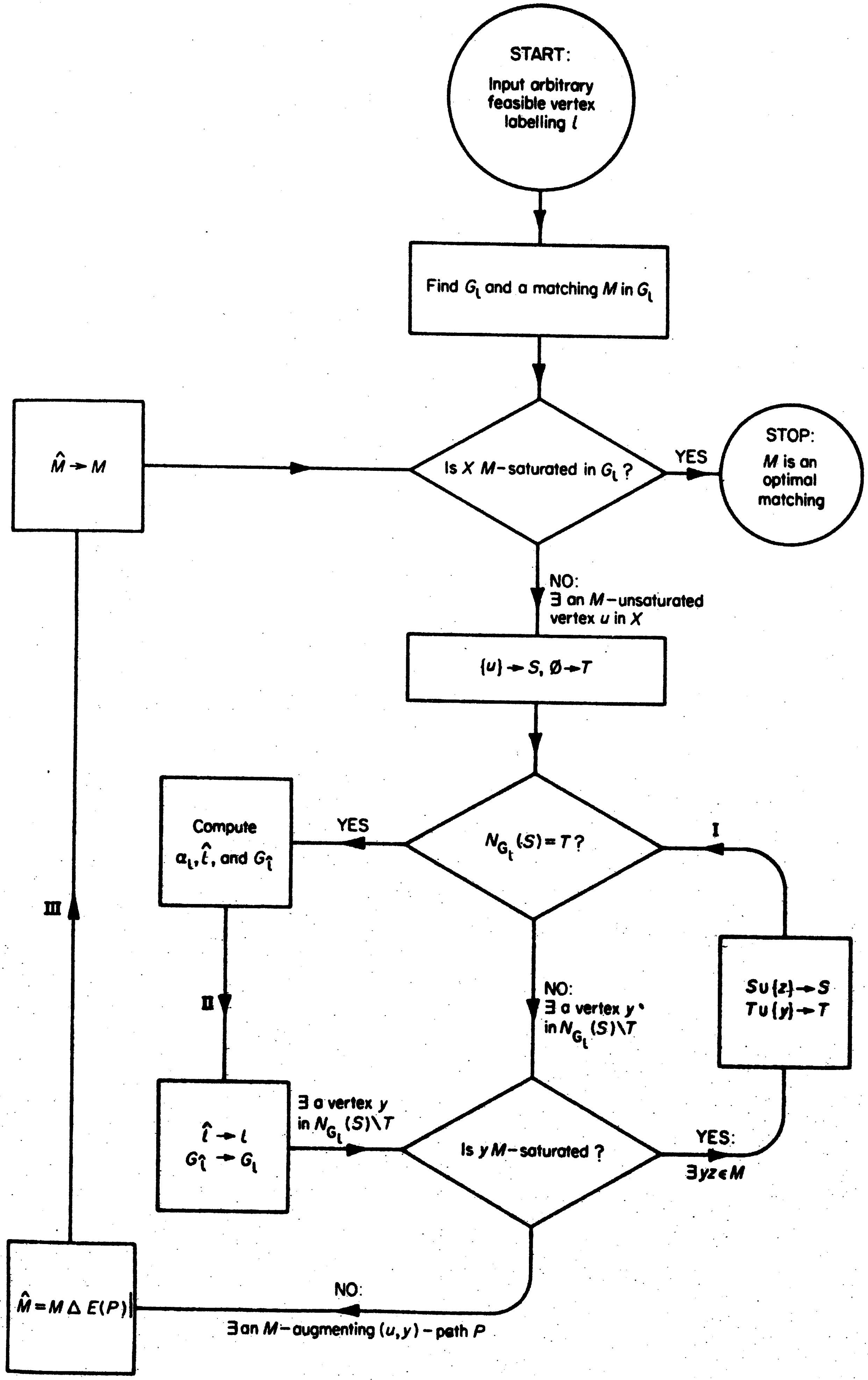


Figure 5.17. The Kuhn–Munkres algorithm

A flow diagram for the Kuhn–Munkres algorithm is given in figure 5.17. In cycle II, the number of computations required to compute $G_{\hat{l}}$ is clearly of order ν^2. Since the algorithm can cycle through I and II at most $|X|$ times before finding an M-augmenting path, and since the initial matching can be augmented at most $|X|$ times before an optimal matching is found, we see that the Kuhn–Munkres algorithm is a good algorithm.

Exercise

5.5.1 A *diagonal* of an $n \times n$ matrix is a set of n entries no two of which belong to the same row or the same column. The weight of a diagonal is the sum of the entries in it. Find a minimum-weight diagonal in the following matrix:

$$\begin{bmatrix} 4 & 5 & 8 & 10 & 11 \\ 7 & 6 & 5 & 7 & 4 \\ 8 & 5 & 12 & 9 & 6 \\ 6 & 6 & 13 & 10 & 7 \\ 4 & 5 & 7 & 9 & 8 \end{bmatrix}$$

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