# 7 Independent Sets and Cliques

## 7.1 INDEPENDENT SETS

A subset $S$ of $V$ is called an *independent set* of $G$ if no two vertices of $S$ are adjacent in $G$. An independent set is *maximum* if $G$ has no independent set $S'$ with $|S'|>|S|$. Examples of independent sets are shown in figure 7.1.

Recall that a subset $K$ of $V$ such that every edge of $G$ has at least one end in $K$ is called a covering of $G$. The two examples of independent sets given in figure 7.1 are both complements of coverings. It is not difficult to see that this is always the case.

*Theorem 7.1* A set $S\subseteq V$ is an independent set of $G$ if and only if $V\backslash S$ is a covering of $G$.

*Proof* By definition, $S$ is an independent set of $G$ if and only if no edge of $G$ has both ends in $S$ or, equivalently, if and only if each edge has at least one end in $V\backslash S$. But this is so if and only if $V\backslash S$ is a covering of $G$ □

The number of vertices in a maximum independent set of $G$ is called the *independence number* of $G$ and is denoted by $\alpha(G)$; similarly, the number of vertices in a minimum covering of $G$ is the *covering number* of $G$ and is denoted by $\beta(G)$.

*Corollary 7.1* $\alpha+\beta=\nu$.

*Proof* Let $S$ be a maximum independent set of $G$, and let $K$ be a minimum covering of $G$. Then, by theorem 7.1, $V\backslash K$ is an independent set

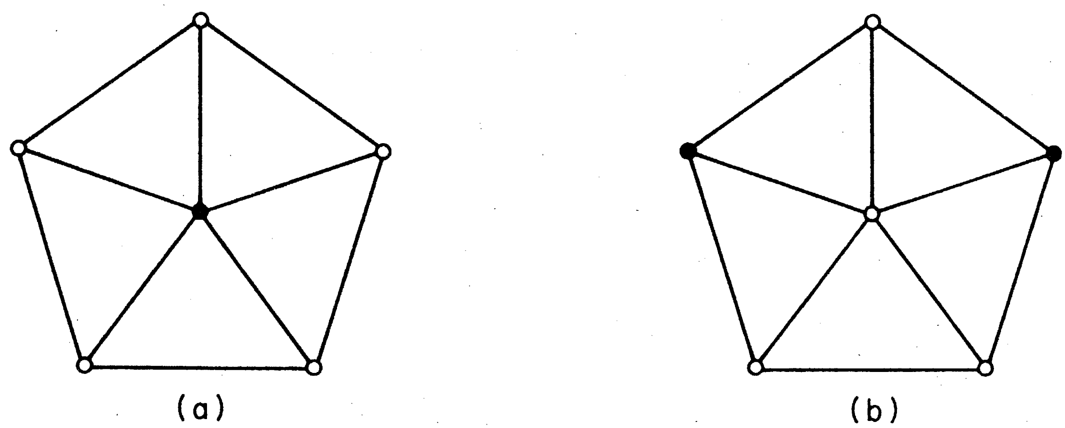


Figure 7.1. (*a*) An independent set; (*b*) a maximum independent set

and $V\backslash S$ is a covering. Therefore

$$\nu-\beta=|V\backslash K|\le\alpha \tag{7.1}$$

and

$$\nu-\alpha=|V\backslash S|\ge\beta \tag{7.2}$$

Combining (7.1) and (7.2) we have $\alpha+\beta=\nu$ □

The edge analogue of an independent set is a set of links no two of which are adjacent, that is, a matching. The edge analogue of a covering is called an edge covering. An *edge covering* of $G$ is a subset $L$ of $E$ such that each vertex of $G$ is an end of some edge in $L$. Note that edge coverings do not always exist; a graph $G$ has an edge covering if and only if $\delta>0$. We denote the number of edges in a maximum matching of $G$ by $\alpha'(G)$, and the number of edges in a minimum edge covering of $G$ by $\beta'(G)$; the numbers $\alpha'(G)$ and $\beta'(G)$ are the *edge independence number* and *edge covering number* of $G$, respectively.

Matchings and edge coverings are not related to one another as simply as are independent sets and coverings; the complement of a matching need not be an edge covering, nor is the complement of an edge covering necessarily a matching. However, it so happens that the parameters $\alpha'$ and $\beta'$ are related in precisely the same manner as are $\alpha$ and $\beta$.

*Theorem* 7.2 (Gallai, 1959) If $\delta>0$, then $\alpha'+\beta'=\nu$.

*Proof* Let $M$ be a maximum matching in $G$ and let $U$ be the set of $M$-unsaturated vertices. Since $\delta>0$ and $M$ is maximum, there exists a set $E'$ of $|U|$ edges, one incident with each vertex in $U$. Clearly, $M\cup E'$ is an edge covering of $G$, and so

$$\beta'\le|M\cup E'|=\alpha'+(\nu-2\alpha')=\nu-\alpha'$$

or

$$\alpha'+\beta'\le\nu \tag{7.3}$$

Now let $L$ be a minimum edge covering of $G$, set $H=G[L]$ and let $M$ be a maximum matching in $H$. Denote the set of $M$-unsaturated vertices in $H$ by $U$. Since $M$ is maximum, $H[U]$ has no links and therefore

$$|L|-|M|=|L\backslash M|\ge|U|=\nu-2\,|M|$$

Because $H$ is a subgraph of $G$, $M$ is a matching in $G$ and so

$$\alpha'+\beta'\ge|M|+|L|\ge\nu \tag{7.4}$$

Combining (7.3) and (7.4), we have $\alpha'+\beta'=\nu$ □

We can now prove a theorem that bears a striking formal resemblance to König's theorem (5.3).

*Theorem 7.3* In a bipartite graph $G$ with $\delta > 0$, the number of vertices in a maximum independent set is equal to the number of edges in a minimum edge covering.

*Proof* Let $G$ be a bipartite graph with $\delta > 0$. By corollary 7.1 and theorem 7.2, we have

$$\alpha + \beta = \alpha' + \beta'$$

and, since $G$ is bipartite, it follows from theorem 5.3 that $\alpha' = \beta$. Thus $\alpha = \beta'$ □

Even though the concept of an independent set is analogous to that of a matching, there exists no theory of independent sets comparable to the theory of matchings presented in chapter 5; for example, no good algorithm for finding a maximum independent set in a graph is known. However, there are two interesting theorems that relate the number of vertices in a maximum independent set of a graph to various other parameters of the graph. These theorems will be discussed in sections 7.2 and 7.3.

*Exercises*

7.1.1 (*a*) Show that $G$ is bipartite if and only if $\alpha(H) \geq \frac{1}{2}\nu(H)$ for every subgraph $H$ of $G$.
(*b*) Show that $G$ is bipartite if and only if $\alpha(H) = \beta'(H)$ for every subgraph $H$ of $G$ such that $\delta(H) > 0$.

7.1.2 A graph is *$\alpha$-critical* if $\alpha(G - e) > \alpha(G)$ for all $e \in E$. Show that a connected $\alpha$-critical graph has no cut vertices.

7.1.3 A graph $G$ is *$\beta$-critical* if $\beta(G - e) < \beta(G)$ for all $e \in E$. Show that
(*a*) a connected $\beta$-critical graph has no cut vertices;
(*b*)* if $G$ is connected, then $\beta \leq \frac{1}{2}(\varepsilon + 1)$.

## 7.2 RAMSEY'S THEOREM

In this section we deal only with simple graphs. A *clique* of a simple graph $G$ is a subset $S$ of $V$ such that $G[S]$ is complete. Clearly, $S$ is a clique of $G$ if and only if $S$ is an independent set of $G^c$, and so the two concepts are complementary.

If $G$ has no large cliques, then one might expect $G$ to have a large independent set. That this is indeed the case was first proved by Ramsey (1930). He showed that, given any positive integers $k$ and $l$, there exists a smallest integer $r(k, l)$ such that every graph on $r(k, l)$ vertices contains either a clique of $k$ vertices or an independent set of $l$ vertices. For example, it is easy to see that

$$r(1, l) = r(k, 1) = 1 \tag{7.5}$$

and

$$r(2, l) = l, \qquad r(k, 2) = k \tag{7.6}$$

The numbers $r(k, l)$ are known as the *Ramsey numbers*. The following theorem on Ramsey numbers is due to Erdös and Szekeres (1935) and Greenwood and Gleason (1955).

*Theorem 7.4* For any two integers $k \geq 2$ and $l \geq 2$

$$r(k, l) \leq r(k, l-1) + r(k-1, l) \tag{7.7}$$

Furthermore, if $r(k, l-1)$ and $r(k-1, l)$ are both even, then strict inequality holds in (7.7).

*Proof* Let $G$ be a graph on $r(k, l-1) + r(k-1, l)$ vertices, and let $v \in V$. We distinguish two cases:

(i) $v$ is nonadjacent to a set $S$ of at least $r(k, l-1)$ vertices, or
(ii) $v$ is adjacent to a set $T$ of at least $r(k-1, l)$ vertices.

Note that either case (i) or case (ii) must hold because the number of vertices to which $v$ is nonadjacent plus the number of vertices to which $v$ is adjacent is equal to $r(k, l-1) + r(k-1, l) - 1$.

In case (i), $G[S]$ contains either a clique of $k$ vertices or an independent set of $l-1$ vertices, and therefore $G[S \cup \{v\}]$ contains either a clique of $k$ vertices or an independent set of $l$ vertices. Similarly, in case (ii), $G[T \cup \{v\}]$ contains either a clique of $k$ vertices or an independent set of $l$ vertices. Since one of case (i) and case (ii) must hold, it follows that $G$ contains either a clique of $k$ vertices or an independent set of $l$ vertices. This proves (7.7).

Now suppose that $r(k, l-1)$ and $r(k-1, l)$ are both even, and let $G$ be a graph on $r(k, l-1) + r(k-1, l) - 1$ vertices. Since $G$ has an odd number of vertices, it follows from corollary 1.1 that some vertex $v$ is of even degree; in particular, $v$ cannot be adjacent to precisely $r(k-1, l) - 1$ vertices. Consequently, either case (i) or case (ii) above holds, and therefore $G$ contains either a clique of $k$ vertices or an independent set of $l$ vertices. Thus

$$r(k, l) \leq r(k, l-1) + r(k-1, l) - 1$$

as stated □

The determination of the Ramsey numbers in general is a very difficult unsolved problem. Lower bounds can be obtained by the construction of suitable graphs. Consider, for example, the four graphs in figure 7.2.

The 5-cycle (figure 7.2*a*) contains no clique of three vertices and no independent set of three vertices. It shows, therefore, that

$$r(3, 3) \geq 6 \tag{7.8}$$

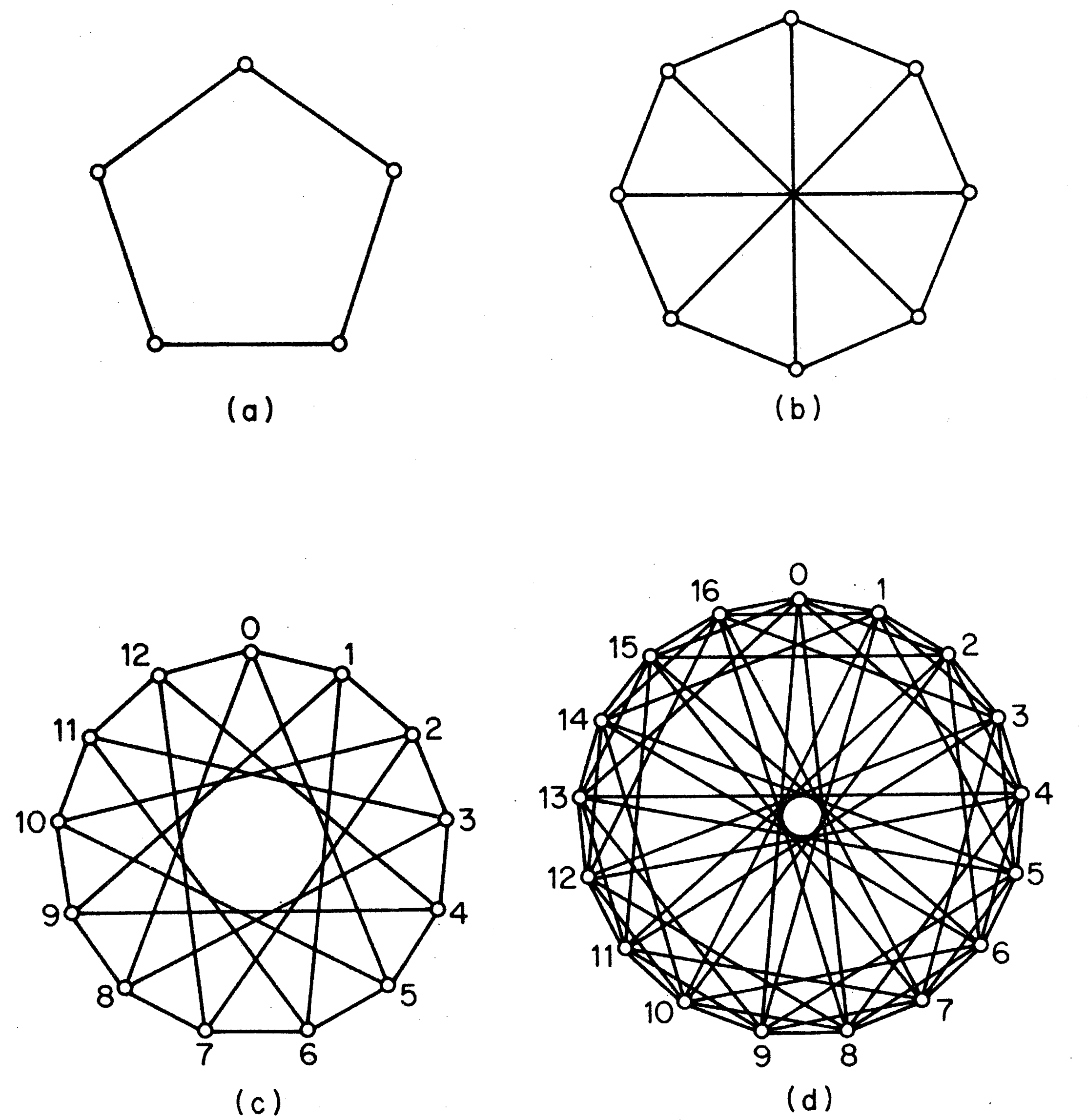



Figure 7.2. (*a*) A (3,3)-Ramsey graph; (*b*) a (3,4)-Ramsey graph; (*c*) a (3,5)-Ramsey graph; (*d*) a (4,4)-Ramsey graph

The graph of figure 7.2*b* contains no clique of three vertices and no independent set of four vertices. Hence

$$r(3,4) \geq 9 \tag{7.9}$$

Similarly, the graph of figure 7.2*c* shows that

$$r(3,5) \geq 14 \tag{7.10}$$

and the graph of figure 7.2*d* yields

$$r(4,4) \geq 18 \tag{7.11}$$

With the aid of theorem 7.4 and equations (7.6) we can now show that equality in fact holds in (7.8), (7.9), (7.10) and (7.11). Firstly, by (7.7) and (7.6)

$$r(3,3) \leq r(3,2) + r(2,3) = 6$$

and therefore, using (7.8), we have $r(3, 3) = 6$. Noting that $r(3, 3)$ and $r(2, 4)$ are both even, we apply theorem 7.4 and (7.6) to obtain

$$r(3, 4) \leq r(3, 3) + r(2, 4) - 1 = 9$$

With (7.9) this gives $r(3, 4) = 9$. Now we again apply (7.7) and (7.6) to obtain

$$r(3, 5) \leq r(3, 4) + r(2, 5) = 14$$

and

$$r(4, 4) \leq r(4, 3) + r(3, 4) = 18$$

which, together with (7.10) and (7.11), respectively, yield $r(3, 5) = 14$ and $r(4, 4) = 18$.

The following table shows all Ramsey numbers $r(k, l)$ known to date.

| $k \backslash l$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3 | 1 | 3 | 6 | 9 | 14 | 18 | 23 |
| 4 | 1 | 4 | 9 | 18 | | | |

A $(k, l)$-*Ramsey graph* is a graph on $r(k, l) - 1$ vertices that contains neither a clique of $k$ vertices nor an independent set of $l$ vertices. By definition of $r(k, l)$ such graphs exist for all $k \geq 2$ and $l \geq 2$. Ramsey graphs often seem to possess interesting structures. All of the graphs in figure 7.2 are Ramsey graphs; the last two can be obtained from finite fields in the following way. We get the (3, 5)-Ramsey graph by regarding the thirteen vertices as elements of the field of integers modulo 13, and joining two vertices by an edge if their difference is a cubic residue of 13 (either 1, 5, 8 or 12); the (4, 4)-Ramsey graph is obtained by regarding the vertices as elements of the field of integers modulo 17, and joining two vertices if their difference is a quadratic residue of 17 (either 1, 2, 4, 8, 9, 13, 15 or 16). It has been conjectured that the $(k, k)$-Ramsey graphs are always self-complementary (that is, isomorphic to their complements); this is true for $k = 2$, 3 and 4.

In general, theorem 7.4 yields the following upper bound for $r(k, l)$.

*Theorem 7.5* $r(k, l) \leq \binom{k+l-2}{k-1}$

*Proof* By induction on $k + l$. Using (7.5) and (7.6) we see that the theorem holds when $k + l \leq 5$. Let $m$ and $n$ be positive integers, and assume that the theorem is valid for all positive integers $k$ and $l$ such that

$5 \le k + l < m + n$. Then, by theorem 7.4 and the induction hypothesis

$$\begin{aligned} r(m, n) &\le r(m, n-1) + r(m-1, n) \\ &\le \binom{m+n-3}{m-1} + \binom{m+n-3}{m-2} = \binom{m+n-2}{m-1} \end{aligned}$$

Thus the theorem holds for all values of $k$ and $l$ □

A lower bound for $r(k, k)$ is given in the next theorem. It is obtained by means of a powerful technique known as the *probabilistic method* (see Erdös and Spencer, 1974). The probabilistic method is essentially a crude counting argument. Although nonconstructive, it can often be applied to assert the existence of a graph with certain specified properties.

*Theorem 7.6* (Erdös, 1947) $r(k, k) \ge 2^{k/2}$

*Proof.* Since $r(1, 1) = 1$ and $r(2, 2) = 2$, we may assume that $k \ge 3$. Denote by $\mathcal{G}_n$ the set of simple graphs with vertex set $\{v_1, v_2, \ldots, v_n\}$, and by $\mathcal{G}_n^k$ the set of those graphs in $\mathcal{G}_n$ that have a clique of $k$ vertices. Clearly

$$|\mathcal{G}_n| = 2^{\binom{n}{2}} \tag{7.12}$$

since each subset of the $\binom{n}{2}$ possible edges $v_iv_j$ determines a graph in $\mathcal{G}_n$. Similarly, the number of graphs in $\mathcal{G}_n$ having a particular set of $k$ vertices as a clique is $2^{\binom{n}{2}-\binom{k}{2}}$. Since there are $\binom{n}{k}$ distinct $k$-element subsets of $\{v_1, v_2, \ldots, v_n\}$, we have

$$|\mathcal{G}_n^k| \le \binom{n}{k} 2^{\binom{n}{2}-\binom{k}{2}} \tag{7.13}$$

By (7.12) and (7.13)

$$\frac{|\mathcal{G}_n^k|}{|\mathcal{G}_n|} \le \binom{n}{k} 2^{-\binom{k}{2}} < \frac{n^k 2^{-\binom{k}{2}}}{k!} \tag{7.14}$$

Suppose, now, that $n < 2^{k/2}$. From (7.14) it follows that

$$\frac{|\mathcal{G}_n^k|}{|\mathcal{G}_n|} < \frac{2^{k^2/2} 2^{-\binom{k}{2}}}{k!} = \frac{2^{k/2}}{k!} < \tfrac{1}{2}$$

Therefore, fewer than half of the graphs in $\mathcal{G}_n$ contain a clique of $k$ vertices. Also, because $\mathcal{G}_n = \{G \mid G^c \in \mathcal{G}_n\}$, fewer than half of the graphs in $\mathcal{G}_n$ contain an independent set of $k$ vertices. Hence some graph in $\mathcal{G}_n$ contains neither a clique of $k$ vertices nor an independent set of $k$ vertices. Because this holds for any $n < 2^{k/2}$, we have $r(k, k) \ge 2^{k/2}$ □

From theorem 7.6 we can immediately deduce a lower bound for $r(k, l)$.

*Corollary 7.6* If $m=\min\{k, l\}$, then $r(k, l)\geq 2^{m/2}$

All known lower bounds for $r(k, l)$ obtained by constructive arguments are much weaker than that given in corollary 7.6; the best is due to Abbott (1972), who shows that $r(2^n+1, 2^n+1)\geq 5^n+1$ (exercise 7.2.4).

The Ramsey numbers $r(k, l)$ are sometimes defined in a slightly different way from that given at the beginning of this section. One easily sees that $r(k, l)$ can be thought of as the smallest integer $n$ such that every 2-edge colouring $(E_1, E_2)$ of $K_n$ contains either a complete subgraph on $k$ vertices, all of whose edges are in colour 1, or a complete subgraph on $l$ vertices, all of whose edges are in colour 2. Expressed in this form, the Ramsey numbers have a natural generalisation. We define $r(k_1, k_2, \ldots, k_m)$ to be the smallest integer $n$ such that every $m$-edge colouring $(E_1, E_2, \ldots, E_m)$ of $K_n$ contains, for some $i$, a complete subgraph on $k_i$ vertices, all of whose edges are in colour $i$.

The following theorem and corollary generalise (7.7) and theorem 7.5, and can be proved in a similar manner. They are left as an exercise (7.2.2).

*Theorem 7.7* $r(k_1, k_2, \ldots, k_m)\leq r(k_1-1, k_2, \ldots, k_m)+$
$r(k_1, k_2-1, \ldots, k_m)+\ldots+r(k_1, k_2, \ldots, k_m-1)-m+2$

*Corollary 7.7* $r(k_1+1, k_2+1, \ldots, k_m+1)\leq\dfrac{(k_1+k_2+\ldots+k_m)!}{k_1!\,k_2!\ldots k_m!}$

*Exercises*

7.2.1 Show that, for all $k$ and $l$, $r(k, l)=r(l, k)$.

7.2.2 Prove theorem 7.7 and corollary 7.7.

**7.2.3** Let $r_n$ denote the Ramsey number $r(k_1, k_2, \ldots, k_n)$ with $k_i=3$ for all $i$.

(*a*) Show that $r_n\leq n(r_{n-1}-1)+2$.
(*b*) Noting that $r_2=6$, use (*a*) to show that $r_n\leq[n!\,e]+1$.
(*c*) Deduce that $r_3\leq 17$.
(Greenwood and Gleason, 1955 have shown that $r_3=17$.)

7.2.4 The *composition* of simple graphs $G$ and $H$ is the simple graph $G[H]$ with vertex set $V(G)\times V(H)$, in which $(u, v)$ is adjacent to $(u', v')$ if and only if either $uu'\in E(G)$ or $u=u'$ and $vv'\in E(H)$.

(*a*) Show that $\alpha(G[H])\leq\alpha(G)\alpha(H)$.
(*b*) Using (*a*), show that

$$r(kl+1, kl+1)-1\geq(r(k+1, k+1)-1)\times(r(l+1, l+1)-1)$$

(*c*) Deduce that $r(2^n+1, 2^n+1)\geq 5^n+1$ for all $n\geq 0$.

(H. L. Abbott)

7.2.5 Show that the join of a 3-cycle and a 5-cycle contains no $K_6$, but that every 2-edge colouring yields a monochromatic triangle.

(R. L. Graham)

(Folkman, 1970 has constructed a graph containing no $K_4$ in which every 2-edge colouring yields a monochromatic triangle—this graph has a very large number of vertices.)

7.2.6 Let $G_1, G_2, \ldots, G_m$ be simple graphs. The *generalised Ramsey number* $r(G_1, G_2, \ldots, G_m)$ is the smallest integer $n$ such that every $m$-edge colouring $(E_1, E_2, \ldots, E_m)$ of $K_n$ contains, for some $i$, a subgraph isomorphic to $G_i$ in colour $i$. Show that

(*a*) if $G$ is a path of length three and $H$ is a 4-cycle, then $r(G, G) = 5$, $r(G, H) = 5$ and $r(H, H) = 6$;

(*b*)* if $T$ is any tree on $m$ vertices and if $m-1$ divides $n-1$, then $r(T, K_{1,n}) = m+n-1$;

(*c*)* if $T$ is any tree on $m$ vertices, then $r(T, K_n) = (m-1)(n-1)+1$.

(V. Chvátal)

## 7.3 TURÁN'S THEOREM

In this section, we shall prove a well-known theorem due to Turán (1941). It determines the maximum number of edges that a simple graph on $\nu$ vertices can have without containing a clique of size $m+1$. Turán's theorem has become the basis of a significant branch of graph theory known as *extremal graph theory* (see Erdös, 1967). We shall derive it from the following result of Erdös (1970).

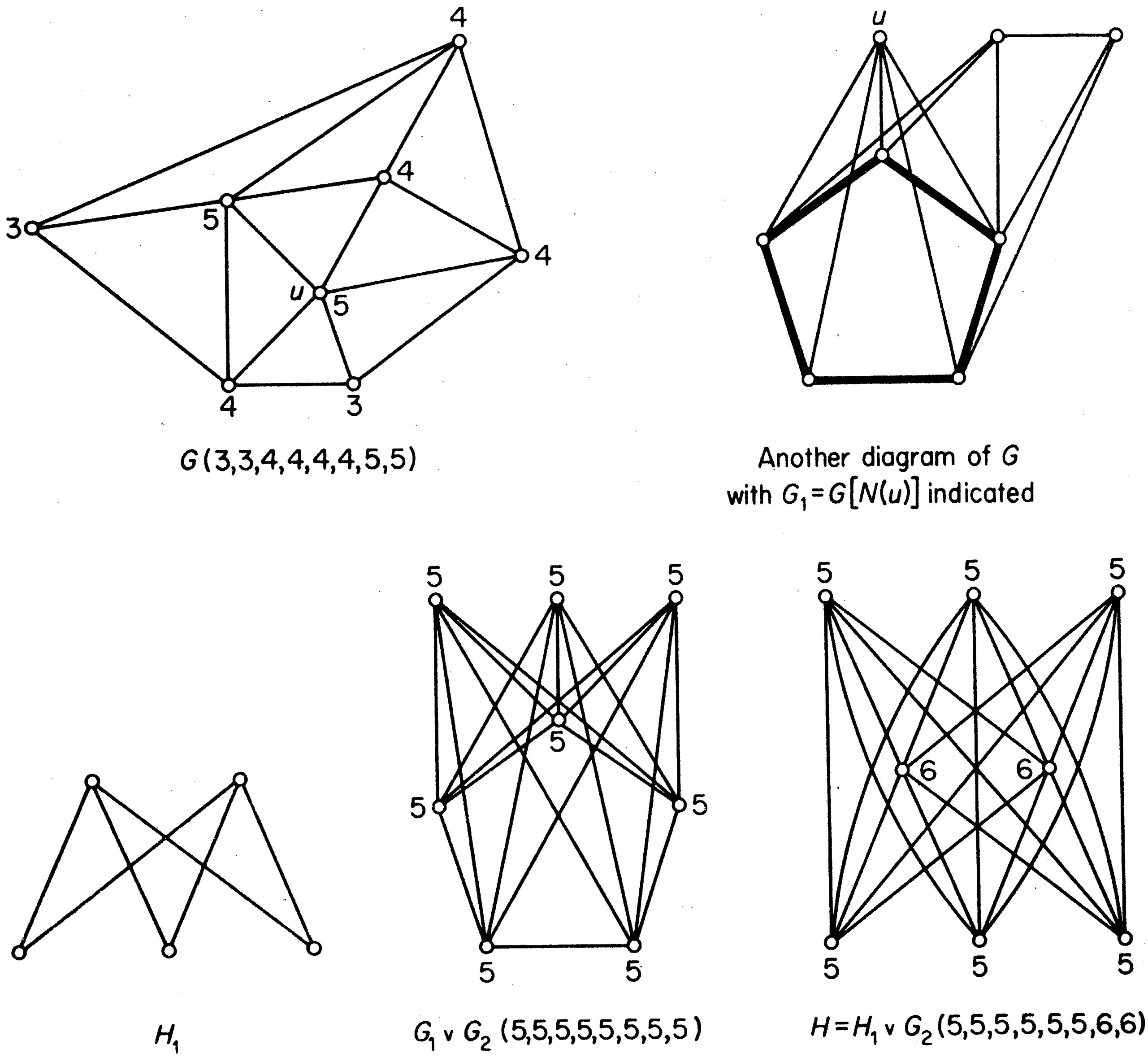
*Theorem 7.8* If a simple graph $G$ contains no $K_{m+1}$, then $G$ is degree-majorised by some complete $m$-partite graph $H$. Moreover, if $G$ has the same degree sequence as $H$, then $G \cong H$.

*Proof* By induction on $m$. The theorem is trivial for $m = 1$. Assume that it holds for all $m < n$, and let $G$ be a simple graph which contains no $K_{n+1}$. Choose a vertex $u$ of degree $\Delta$ in $G$, and set $G_1 = G[N(u)]$. Since $G$ contains no $K_{n+1}$, $G_1$ contains no $K_n$ and therefore, by the induction hypothesis, is degree-majorised by some complete $(n-1)$-partite graph $H_1$.

Next, set $V_1 = N(u)$ and $V_2 = V \backslash V_1$, and denote by $G_2$ the graph whose vertex set is $V_2$ and whose edge set is empty. Consider the join $G_1 \vee G_2$ of $G_1$ and $G_2$. Since

$$N_G(v) \subseteq N_{G_1 \vee G_2}(v) \quad \text{for} \quad v \in V_1 \tag{7.15}$$

and since each vertex of $V_2$ has degree $\Delta$ in $G_1 \vee G_2$, $G$ is degree-majorised by $G_1 \vee G_2$. Therefore $G$ is also degree-majorised by the complete $n$-partite graph $H = H_1 \vee G_2$. (See figure 7.3 for illustration.)



$G(3,3,4,4,4,4,5,5)$

Another diagram of $G$ with $G_1 = G[N(u)]$ indicated

$H_1$

$G_1 \vee G_2$ (5,5,5,5,5,5,5,5)

$H = H_1 \vee G_2$ (5,5,5,5,5,5,6,6)

Figure 7.3

Suppose, now, that $G$ has the same degree sequence as $H$. Then $G$ has the same degree sequence as $G_1 \vee G_2$ and hence equality must hold in (7.15). Thus, in $G$, every vertex of $V_1$ must be joined to every vertex of $V_2$. It follows that $G = G_1 \vee G_2$. Since $G = G_1 \vee G_2$ has the same degree sequence as $H = H_1 \vee G_2$, the graphs $G_1$ and $H_1$ must have the same degree sequence and therefore, by the induction hypothesis, be isomorphic. We conclude that $G \cong H$ □

It is interesting to note that the above theorem bears a striking similarity to theorem 4.6.

Let $T_{m,n}$ denote the complete $m$-partite graph on $n$ vertices in which all parts are as equal in size as possible; the graph $H$ of figure 7.3 is $T_{3,8}$.

*Theorem* 7.9 If $G$ is simple and contains no $K_{m+1}$, then $\varepsilon(G) \leq \varepsilon(T_{m,\nu})$. Moreover, $\varepsilon(G) = \varepsilon(T_{m,\nu})$ only if $G \cong T_{m,\nu}$.

*Proof* Let $G$ be a simple graph that contains no $K_{m+1}$. By theorem 7.8, $G$ is degree-majorised by some complete $m$-partite graph $H$. It follows from theorem 1.1 that

$$\varepsilon(G) \leq \varepsilon(H) \tag{7.16}$$

But (exercise 1.2.9)

$$\varepsilon(H) \leq \varepsilon(T_{m,\nu}) \tag{7.17}$$

Therefore, from (7.16) and (7.17)

$$\varepsilon(G) \leq \varepsilon(T_{m,\nu}) \tag{7.18}$$

proving the first assertion.

Suppose, now, that equality holds in (7.18). Then equality must hold in both (7.16) and (7.17). Since $\varepsilon(G) = \varepsilon(H)$ and $G$ is degree-majorised by $H$, $G$ must have the same degree sequence as $H$. Therefore, by theorem 7.8, $G \cong H$. Also, since $\varepsilon(H) = \varepsilon(T_{m,\nu})$, it follows (exercise 1.2.9) that $H \cong T_{m,\nu}$. We conclude that $G \cong T_{m,\nu}$ □

*Exercises*

7.3.1 In a group of nine people, one person knows two of the others, two people each know four others, four each know five others, and the remaining two each know six others. Show that there are three people who all know one another.

7.3.2 A certain bridge club has a special rule to the effect that four members may play together only if no two of them have previously partnered one another. At one meeting fourteen members, each of whom has previously partnered five others, turn up. Three games are played, and then proceedings come to a halt because of the club rule. Just as the members are preparing to leave, a new member, unknown to any of them, arrives. Show that at least one more game can now be played.

7.3.3 (*a*) Show that if $G$ is simple and $\varepsilon > \nu^2/4$, then $G$ contains a triangle.

(*b*) Find a simple graph $G$ with $\varepsilon = [\nu^2/4]$ that contains no triangle.

(*c*)* Show that if $G$ is simple and not bipartite with $\varepsilon > ((\nu-1)^2/4)+1$, then $G$ contains a triangle.

(*d*) Find a simple non-bipartite graph $G$ with $\varepsilon = [(\nu-1)^2/4]+1$ that contains no triangle. (P. Erdös)

7.3.4 (*a*)* Show that if $G$ is simple and $\sum_{v\in V}\binom{d(v)}{2} > (m-1)\binom{\nu}{2}$, then $G$ contains $K_{2,m}$ $(m \geq 2)$.

(*b*) Deduce that if $G$ is simple and $\varepsilon > \frac{(m-1)^{\frac{1}{2}}\nu^{\frac{3}{2}}}{2} + \frac{\nu}{4}$, then $G$ contains $K_{2,m}$ $(m \geq 2)$.

(*c*) Show that, given a set of $n$ points in the plane, the number of pairs of points at distance exactly 1 is at most $n^{\frac{3}{2}}/\sqrt{2}+n/4$.

7.3.5 Show that if G is simple and $\varepsilon > \dfrac{(m-1)^{1/m}\nu^{2-1/m}}{2}+\dfrac{(m-1)\nu}{2}$ then G contains $K_{m,m}$.

## APPLICATIONS

### 7.4 SCHUR'S THEOREM

Consider the partition $(\{1, 4, 10, 13\}, \{2, 3, 11, 12\}, \{5, 6, 7, 8, 9\})$ of the set of integers $\{1, 2, \ldots, 13\}$. We observe that in no subset of the partition are there integers $x$, $y$ and $z$ (not necessarily distinct) which satisfy the equation

$$x+y=z \tag{7.19}$$

Yet, no matter how we partition $\{1, 2, \ldots, 14\}$ into three subsets, there always exists a subset of the partition which contains a solution to (7.19). Schur (1916) proved that, in general, given any positive integer $n$, there exists an integer $f_n$ such that, in any partition of $\{1, 2, \ldots, f_n\}$ into $n$ subsets, there is a subset which contains a solution to (7.19). We shall show how Schur's theorem follows from the existence of the Ramsey numbers $r_n$ (defined in exercise 7.2.3).

*Theorem 7.10* Let $(S_1, S_2, \ldots, S_n)$ be any partition of the set of integers $\{1, 2, \ldots, r_n\}$. Then, for some $i$, $S_i$ contains three integers $x$, $y$ and $z$ satisfying the equation $x+y=z$.

*Proof* Consider the complete graph whose vertex set is $\{1, 2, \ldots, r_n\}$. Colour the edges of this graph in colours $1, 2, \ldots, n$ by the rule that the edge $uv$ is assigned colour $j$ if and only if $|u-v| \in S_j$. By Ramsey's theorem (7.7) there exists a monochromatic triangle; that is, there are three vertices $a$, $b$ and $c$ such that $ab$, $bc$ and $ca$ have the same colour, say $i$. Assume, without loss of generality that $a>b>c$ and write $x=a-b$, $y=b-c$ and $z=a-c$. Then $x, y, z \in S_i$ and $x+y=z$ □

Let $s_n$ denote the least integer such that, in any partition of $\{1, 2, \ldots, s_n\}$ into $n$ subsets, there is a subset which contains a solution to (7.19). It can be easily seen that $s_1=2$, $s_2=5$ and $s_3=14$ (exercise 7.4.1). Also, from theorem 7.10 and exercise 7.2.3 we have the upper bound

$$s_n \leq r_n \leq [n!\, e]+1$$

Exercise 7.4.2*b* provides a lower bound for $s_n$.

*Exercises*

7.4.1 Show that $s_1 = 2$, $s_2 = 5$ and $s_3 = 14$.

7.4.2 (*a*) Show that $s_n \geq 3s_{n-1} - 1$.

(*b*) Using (*a*) and the fact that $s_3 = 14$, show that $s_n \geq \frac{1}{2}(27(3)^{n-3} + 1)$. (A better lower bound has been obtained by Abbott and Moser, 1966.)

## 7.5 A GEOMETRY PROBLEM

The *diameter* of a set $S$ of points in the plane is the maximum distance between two points of $S$. It should be noted that this is a purely geometric notion and is quite unrelated to the graph-theoretic concepts of diameter and distance.

We shall discuss sets of diameter 1. A set of $n$ points determines $\binom{n}{2}$ distances between pairs of these points. It is intuitively clear that if $n$ is 'large', then some of these distances must be 'small'. Therefore, for any $d$ between 0 and 1, we can ask how many pairs of points in a set $\{x_1, x_2, \ldots, x_n\}$ of diameter 1 can be at distance greater than $d$. Here, we shall present a solution to one special case of this problem, namely when $d = 1/\sqrt{2}$.

As an illustration, consider the case $n = 6$. We then have six points $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ and $x_6$. If we place them at the vertices of a regular hexagon so that the pairs $(x_1, x_4)$, $(x_2, x_5)$ and $(x_3, x_6)$ are at distance 1, as shown in figure 7.4*a*, these six points constitute a set of diameter 1.

It is easily calculated that the pairs $(x_1, x_2)$, $(x_2, x_3)$, $(x_3, x_4)$, $(x_4, x_5)$, $(x_5, x_6)$ and $(x_6, x_1)$ are at distance 1/2, and the pairs $(x_1, x_3)$, $(x_2, x_4)$, $(x_3, x_5)$, $(x_4, x_6)$, $(x_5, x_1)$ and $(x_6, x_2)$ are at distance $\sqrt{3}/2$. Since $\sqrt{3}/2 > \sqrt{2}/2 = 1/\sqrt{2}$, there are nine pairs of points at distance greater than $1/\sqrt{2}$ in this set of diameter 1.

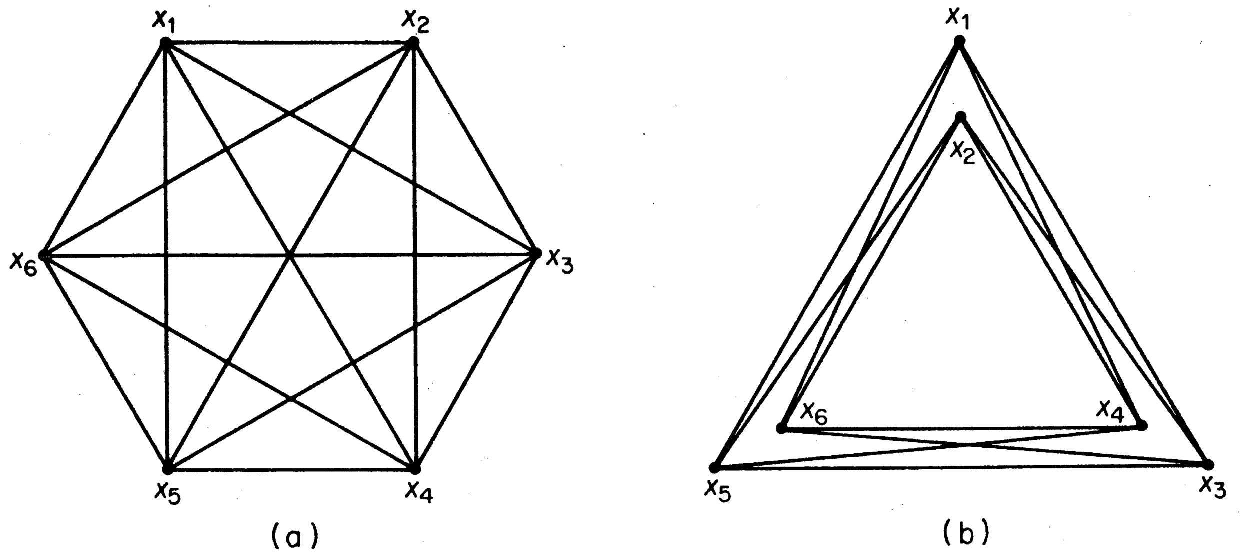



Figure 7.4

However, nine is not the best that we can do with six points. By placing the points in the configuration shown in figure 7.4*b*, all pairs of points except $(x_1, x_2)$, $(x_3, x_4)$ and $(x_5, x_6)$ are at distance greater than $1/\sqrt{2}$. Thus we have twelve pairs at distance greater than $1/\sqrt{2}$; this is, in fact, the best we can do. The solution to the problem in general is given by the following theorem.

*Theorem 7.11* If $\{x_1, x_2, \ldots, x_n\}$ is a set of diameter 1 in the plane, the maximum possible number of pairs of points at distance greater than $1/\sqrt{2}$ is $[n^2/3]$. Moreover, for each $n$, there is a set $\{x_1, x_2, \ldots, x_n\}$ of diameter 1 with exactly $[n^2/3]$ pairs of points at distance greater than $1/\sqrt{2}$.

*Proof* Let $G$ be the graph defined by

$$V(G) = \{x_1, x_2, \ldots, x_n\}$$

and

$$E(G) = \{x_ix_j \mid d(x_i, x_j) > 1/\sqrt{2}\}$$

where $d(x_i, x_j)$ here denotes the *euclidean* distance between $x_i$ and $x_j$. We shall show that $G$ cannot contain a $K_4$.

First, note that any four points in the plane must determine an angle of at least 90°. For the convex hull of the points is either (a) a line, (b) a triangle, or (c) a quadrilateral (see figure 7.5). Clearly, in each case there is an angle $x_ix_jx_k$ of at least 90°.

Now look at the three points $x_i$, $x_j$, $x_k$ which determine this angle. Not all the distances $d(x_i, x_j)$, $d(x_i, x_k)$ and $d(x_j, x_k)$ can be greater than $1/\sqrt{2}$ and less than or equal to 1. For, if $d(x_i, x_j) > 1/\sqrt{2}$ and $d(x_j, x_k) > 1/\sqrt{2}$, then $d(x_i, x_k) > 1$. Since the set $\{x_1, x_2, \ldots, x_n\}$ is assumed to have diameter 1, it follows that, of any four points in $G$, at least one pair cannot be joined by an edge, and hence that $G$ cannot contain a $K_4$. By Turán's theorem (7.9)

$$\varepsilon(G) \leq \varepsilon(T_{3,n}) = [n^2/3]$$

One can construct a set $\{x_1, x_2, \ldots, x_n\}$ of diameter 1 in which exactly

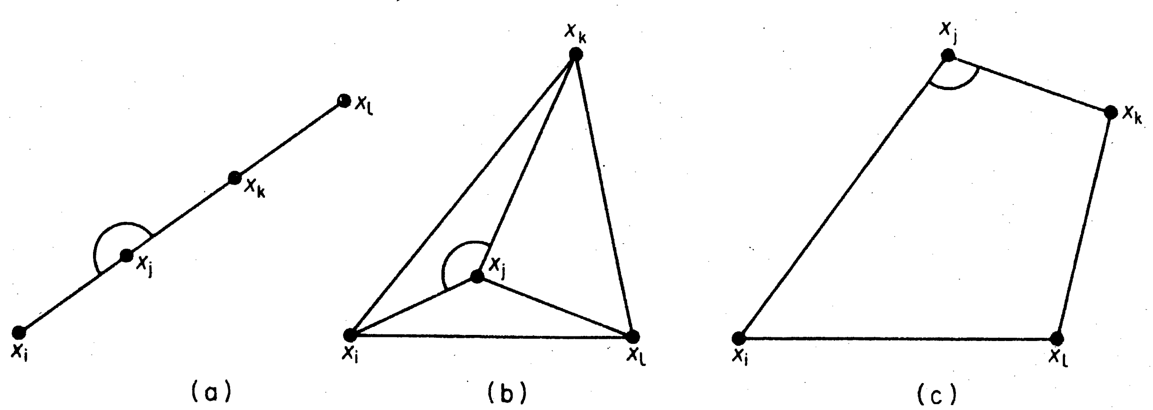



Figure 7.5

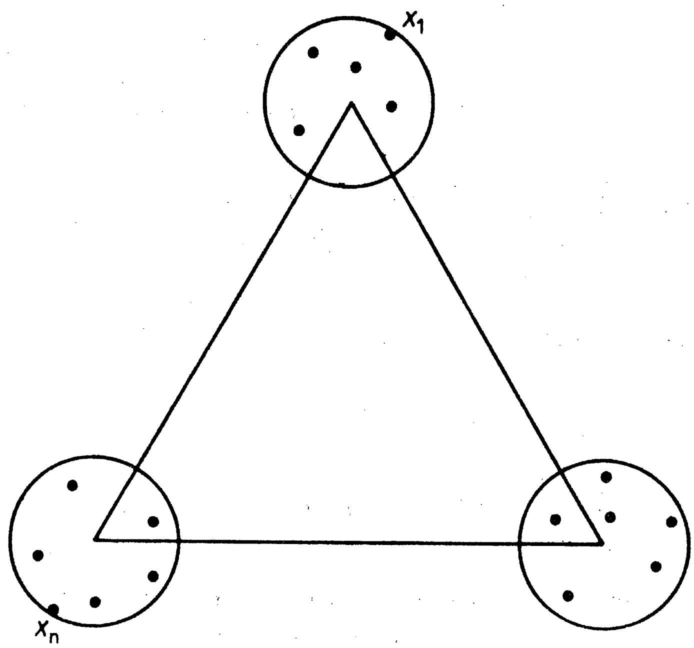



Figure 7.6

$[n^2/3]$ pairs of points are at distance greater than $1/\sqrt{2}$ as follows. Choose $r$ such that $0<r<(1-1/\sqrt{2})/4$, and draw three circles of radius $r$ whose centres are at a distance of $1-2r$ from one another (figure 7.6). Place $x_1, \ldots, x_{[n/3]}$ in one circle, $x_{[n/3]+1}, \ldots, x_{[2n/3]}$ in another, and $x_{[2n/3]+1}, \ldots, x_n$ in the third, in such a way that $d(x_1, x_n)=1$. This set clearly has diameter 1. Also, $d(x_i, x_j)>1/\sqrt{2}$ if and only if $x_i$ and $x_j$ are in different circles, and so there are exactly $[n^2/3]$ pairs $(x_i, x_j)$ for which $d(x_i, x_j)>1/\sqrt{2}$ □

*Exercises*

7.5.1* Let $\{x_1, x_2, \ldots, x_n\}$ be a set of diameter 1 in the plane.

(*a*) Show that the maximum possible number of pairs of points at distance 1 is $n$.

(*b*) Construct a set $\{x_1, x_2, \ldots, x_n\}$ of diameter 1 in the plane in which exactly $n$ pairs of points are at distance 1. (E. Pannwitz)

7.5.2 A flat circular city of radius six miles is patrolled by eighteen police cars, which communicate with one another by radio. If the range of a radio is nine miles, show that, at any time, there are always at least two cars each of which can communicate with at least five other cars.

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