8 Vertex Colourings

8.1 CHROMATIC NUMBER

In chapter 6 we studied edge colourings of graphs. We now turn our attention to the analogous concept of vertex colouring.

A *k-vertex colouring* of G is an assignment of k colours, $1, 2, \ldots, k$, to the vertices of G; the colouring is *proper* if no two distinct adjacent vertices have the same colour. Thus a proper k-vertex colouring of a loopless graph G is a partition $(V_1, V_2, \ldots, V_k)$ of V into k (possibly empty) independent sets. G is *k-vertex-colourable* if G has a proper k-vertex colouring. It will be convenient to refer to a 'proper vertex colouring' as, simply, a *colouring* and to a 'proper k-vertex colouring' as a *k-colouring*; we shall similarly abbreviate 'k-vertex-colourable' to *k-colourable*. Clearly, a graph is k-colourable if and only if its underlying simple graph is k-colourable. Therefore, in discussing colourings, we shall restrict ourselves to simple graphs; a simple graph is 1-colourable if and only if it is empty, and 2-colourable if and only if it is bipartite. The *chromatic number*, $\chi(G)$, of G is the minimum k for which G is k-colourable; if $\chi(G) = k$, G is said to be *k-chromatic*. A 3-chromatic graph is shown in figure 8.1. It has the indicated 3-colouring, and is not 2-colourable since it is not bipartite.

It is helpful, when dealing with colourings, to study the properties of a special class of graphs called critical graphs. We say that a graph G is *critical* if $\chi(H) < \chi(G)$ for every proper subgraph H of G. Such graphs were first investigated by Dirac (1952). A *k-critical* graph is one that is k-chromatic and critical; every k-chromatic graph has a k-critical subgraph. A 4-critical graph, due to Grötzsch (1958), is shown in figure 8.2.

An easy consequence of the definition is that every critical graph is connected. The following theorems establish some of the basic properties of critical graphs.

Theorem 8.1 If G is k-critical, then $\delta \geq k - 1$.

Proof By contradiction. If possible, let G be a k-critical graph with $\delta < k - 1$, and let v be a vertex of degree δ in G. Since G is k-critical, $G - v$ is $(k-1)$-colourable. Let $(V_1, V_2, \ldots, V_{k-1})$ be a $(k-1)$-colouring of $G - v$. By definition, v is adjacent in G to $\delta < k - 1$ vertices, and therefore v must be nonadjacent in G to every vertex of some V_j. But then $(V_1, V_2, \ldots, V_j \cup \{v\}, \ldots, V_{k-1})$ is a $(k-1)$-colouring of G, a contradiction. Thus $\delta \geq k - 1$ □

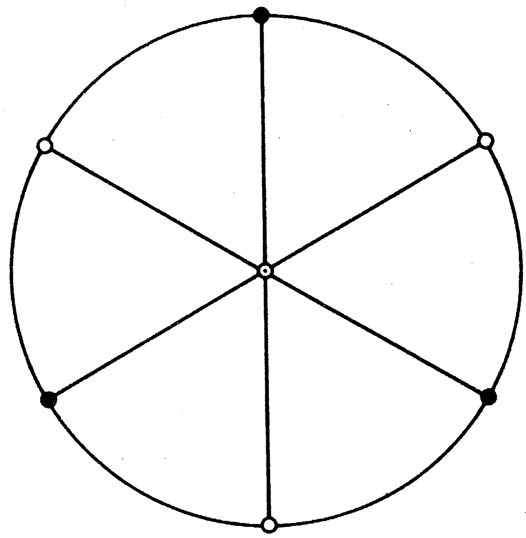


Figure 8.1. A 3-chromatic graph

Corollary 8.1.1 Every k-chromatic graph has at least k vertices of degree at least $k-1$.

Proof Let G be a k-chromatic graph, and let H be a k-critical subgraph of G. By theorem 8.1, each vertex of H has degree at least $k-1$ in H, and hence also in G. The corollary now follows since H, being k-chromatic, clearly has at least k vertices □

Corollary 8.1.2 For any graph G,

$$\chi \leq \Delta + 1$$

Proof This is an immediate consequence of corollary 8.1.1 □

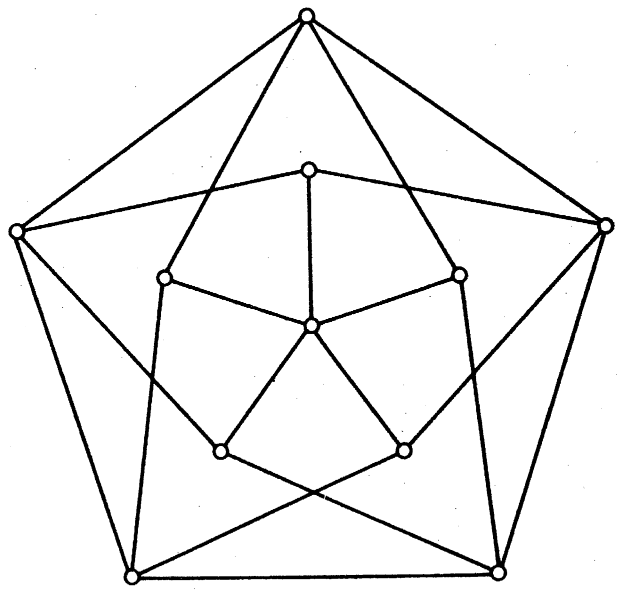


Figure 8.2. The Grötzsch graph—a 4-critical graph

Let S be a vertex cut of a connected graph G, and let the components of $G-S$ have vertex sets $V_1, V_2, \ldots, V_n$. Then the subgraphs $G_i = G[V \cup S]$ are called the S-*components* of G (see figure 8.3). We say that colourings of $G_1, G_2, \ldots, G_n$ *agree* on S if, for every $v \in S$, vertex v is assigned the same colour in each of the colourings.

Theorem 8.2 In a critical graph, no vertex cut is a clique.

Proof By contradiction. Let G be a k-critical graph, and suppose that G has a vertex cut S that is a clique. Denote the S-components of G b $G_1, G_2, \ldots, G_n$. Since G is k-critical, each G_i is $(k-1)$-colourable. Furth-more, because S is a clique, the vertices in S must receive distinct colours in any $(k-1)$-colouring of G_i. It follows that there are $(k-1)$-colourings of $G_1, G_2, \ldots, G_n$ which agree on S. But these colourings together yield a $(k-1)$-colouring of G, a contradiction □

Corollary 8.2 Every critical graph is a block.

Proof If v is a cut vertex, then $\{v\}$ is a vertex cut which is also, trivially, a clique. It follows from theorem 8.2 that no critical graph has a cut vertex; equivalently, every critical graph is a block □

Another consequence of theorem 8.2 is that if a k-critical graph G has a 2-vertex cut $\{u, v\}$, then u and v cannot be adjacent. We shall say that a $\{u, v\}$-component G_i of G is of *type 1* if every $(k-1)$-colouring of G_i assigns the same colour to u and v, and of *type 2* if every $(k-1)$-colouring of G_i assigns different colours to u and v (see figure 8.4).

Theorem 8.3 (Dirac, 1953) Let G be a k-critical graph with a 2-vertex cut $\{u, v\}$. Then

(i) $G = G_1 \cup G_2$, where G_i is a $\{u, v\}$-component of type i $(i = 1, 2)$, and

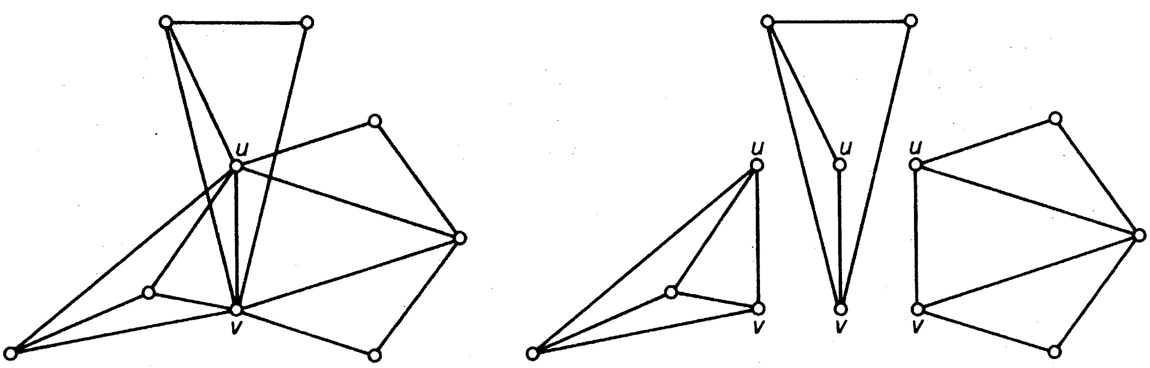


Figure 8.3. (*a*) G; (*b*) the $\{u, v\}$-components of G

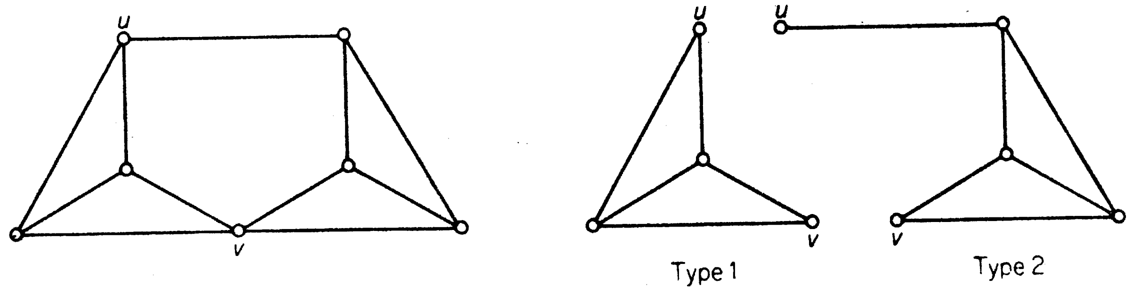


Figure 8.4

(ii) both G_1+uv and $G_2 \cdot uv$ are k-critical (where $G_2 \cdot uv$ denotes the graph obtained from G_2 by identifying u and v).,

Proof (i) Since G is critical, each $\{u, v\}$-component of G is $(k-1)$-colourable. Now there cannot exist $(k-1)$-colourings of these $\{u, v\}$-components all of which agree on $\{u, v\}$, since such colourings would together yield a $(k-1)$-colouring of G. Therefore there are two $\{u, v\}$-components G_1 and G_2 such that no $(k-1)$-colouring of G_1 agrees with any $(k-1)$-colouring of G_2. Clearly one, say G_1, must be of type 1 and the other, G_2, of type 2. Since G_1 and G_2 are of different types, the subgraph $G_1 \cup G_2$ of G is not $(k-1)$-colourable. Therefore, because G is critical, we must have $G = G_1 \cup G_2$.

(ii) Set $H_1 = G_1+uv$. Since G_1 is of type 1, H_1 is k-chromatic. We shall prove that H_1 is critical by showing that, for every edge e of H_1, H_1-e is $(k-1)$-colourable. This is clearly so if $e=uv$, since then $H_1-e=G_1$. Let e be some other edge of H_1. In any $(k-1)$-colouring of $G-e$, the vertices u and v must receive different colours, since G_2 is a subgraph of $G-e$. The restriction of such a colouring to the vertices of G_1 is a $(k-1)$-colouring of H_1-e. Thus G_1+uv is k-critical. An analogous argument shows that $G_2 \cdot uv$ is k-critical □

Corollary 8.3 Let G be a k-critical graph with a 2-vertex cut $\{u, v\}$. Then

$$d(u)+d(v) \geq 3k-5 \quad (8.1)$$

Proof Let G_1 be the $\{u, v\}$-component of type 1 and G_2 the $\{u, v\}$-component of type 2. Set $H_1 = G_1+uv$ and $H_2 = G_2 \cdot uv$. By theorems 8.3 and 8.1

$$d_{H_1}(u)+d_{H_1}(v) \geq 2k-2$$

and

$$d_{H_2}(w) \geq k-1$$

where w is the new vertex obtained by identifying u and v.
It follows that

$$d_{G_1}(u)+d_{G_1}(v) \geq 2k-4$$

and

$$d_{G_2}(u)+d_{G_2}(v)\geq k-1$$

These two inequalities yield (8.1) □

Exercises

8.1.1 Show that if G is simple, then $\chi\geq\nu^2/(\nu^2-2\varepsilon)$.

8.1.2 Show that if any two odd cycles of G have a vertex in common, then $\chi\leq 5$.

8.1.3 Show that if G has degree sequence $(d_1, d_2, \ldots, d_\nu)$ with $d_1\geq d_2\geq \ldots\geq d_\nu$, then $\chi\leq\max_i\min\{d_i+1, i\}$.

(D. J. A. Welsh and M. B. Powell)

8.1.4 Using exercise 8.1.3, show that

(*a*) $\chi\leq\{(2\varepsilon)^{\frac{1}{2}}\}$;

(*b*) $\chi(G)+\chi(G^c)\leq\nu+1$. (E. A. Nordhaus and J. W. Gaddum)

8.1.5 Show that $\chi(G)\leq 1+\max\delta(H)$, where the maximum is taken over all induced subgraphs H of G. (G. Szekeres and H. S. Wilf)

8.1.6* If a k-chromatic graph G has a colouring in which each colour is assigned to at least two vertices, show that G has a k-colouring of this type. (T. Gallai)

8.1.7 Show that the only 1-critical graph is K_1, the only 2-critical graph is K_2, and the only 3-critical graphs are the odd k-cycles with $k\geq 3$.

8.1.8 A graph G is *uniquely k-colourable* if any two k-colourings of G induce the same partition of V. Show that no vertex cut of a k-critical graph induces a uniquely $(k-1)$-colourable subgraph.

8.1.9 (*a*) Show that if u and v are two vertices of a critical graph G, then $N(u)\not\subseteq N(v)$.

(*b*) Deduce that no k-critical graph has exactly $k+1$ vertices.

8.1.10 Show that

(*a*) $\chi(G_1\vee G_2)=\chi(G_1)+\chi(G_2)$;

(*b*) $G_1\vee G_2$ is critical if and only if both G_1 and G_2 are critical.

8.1.11 Let G_1 and G_2 be two k-critical graphs with exactly one vertex v in common, and let vv_1 and vv_2 be edges of G_1 and G_2. Show that the graph $(G_1-vv_1)\cup(G_2-vv_2)+v_1v_2$ is k-critical. (G. Hajós)

8.1.12 For $n=4$ and all $n\geq 6$, construct a 4-critical graph on n vertices.

8.1.13 (*a*)* Let (X, Y) be a partition of V such that $G[X]$ and $G[Y]$ are both n-colourable. Show that, if the edge cut $[X, Y]$ has at most $n-1$ edges, then G is also n-colourable.

(P. C. Kainen)

(*b*) Deduce that every k-critical graph is $(k-1)$-edge-connected.

(G. A. Dirac)

8.2 BROOKS' THEOREM

The upper bound on chromatic number given in corollary 8.1.2 is sometimes very much greater than the actual value. For example, bipartite graphs are 2-chromatic, but can have arbitrarily large maximum degree. In this sense corollary 8.1.2 is a considerably weaker result than Vizing's theorem (6.2). There is another sense in which Vizing's result is stronger. Many graphs G satisfy $\chi' = \Delta + 1$ (see exercises 6.2.2 and 6.2.3). However, as is shown in the following theorem due to Brooks (1941), there are only two types of graph G for which $\chi = \Delta + 1$. The proof of Brooks' theorem given here is by Lovász (1973).

Theorem 8.4 If G is a connected simple graph and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

Proof Let G be a k-chromatic graph which satisfies the hypothesis of the theorem. Without loss of generality, we may assume that G is k-critical. By corollary 8.2, G is a block. Also, since 1-critical and 2-critical graphs are complete and 3-critical graphs are odd cycles (exercise 8.1.7), we have $k \geq 4$.

If G has a 2-vertex cut $\{u, v\}$, corollary 8.3 gives

$$2\Delta \geq d(u) + d(v) \geq 3k - 5 \geq 2k - 1$$

This implies that $\chi = k \leq \Delta$, since 2Δ is even.

Assume, then, that G is 3-connected. Since G is not complete, there are three vertices u, v and w in G such that $uv, vw \in E$ and $uw \notin E$ (exercise 1.6.14). Set $u = v_1$ and $w = v_2$ and let $v_3, v_4, \ldots, v_\nu = v$ be any ordering of the vertices of $G - \{u, w\}$ such that each v_i is adjacent to some v_j with $j > i$. (This can be achieved by arranging the vertices of $G - \{u, w\}$ in nonincreasing order of their distance from v.) We can now describe a Δ-colouring of G: assign colour 1 to $v_1 = u$ and $v_2 = w$; then successively colour $v_3, v_4, \ldots, v_\nu$, each with the first available colour in the list $1, 2, \ldots, \Delta$. By the construction of the sequence $v_1, v_2, \ldots, v_\nu$, each vertex v_i, $1 \leq i \leq \nu - 1$, is adjacent to some vertex v_j with $j > i$, and therefore to at most $\Delta - 1$ vertices v_j with $j < i$. It follows that, when its turn comes to be coloured, v_i is adjacent to at most $\Delta - 1$ colours, and thus that one of the colours $1, 2, \ldots, \Delta$ will be available. Finally, since v_ν is adjacent to two vertices of colour 1 (namely v_1 and v_2), it is adjacent to at most $\Delta - 2$ other colours and can be assigned one of the colours $2, 3, \ldots, \Delta$ □

Exercises

8.2.1 Show that Brooks' theorem is equivalent to the following statement: if G is k-critical ($k \geq 4$) and not complete, then $2\varepsilon \geq \nu(k-1) + 1$.

8.2.2 Use Brooks' theorem to show that if G is loopless with $\Delta = 3$, then $\chi' \leq 4$.

8.3 HAJÓS' CONJECTURE

A *subdivision* of a graph G is a graph that can be obtained from G by a sequence of edge subdivisions. A subdivision of K_4 is shown in figure 8.5. Although no necessary and sufficient condition for a graph to be k-chromatic is known when $k \geq 3$, a plausible necessary condition has been proposed by Hajós (1961): if G is k-chromatic, then G contains a subdivision of K_k. This is known as *Hajós' conjecture.* It should be noted that the condition is not sufficient; for example, a 4-cycle is a subdivision of K_3, but is not 3-chromatic.

For $k = 1$ and $k = 2$, the validity of Hajós' conjecture is obvious. It is also easily verified for $k = 3$, because a 3-chromatic graph necessarily contains an odd cycle, and every odd cycle is a subdivision of K_3. Dirac (1952) settled the case $k = 4$.

Theorem 8.5 If G is 4-chromatic, then G contains a subdivision of K_4.

Proof Let G be a 4-chromatic graph. Note that if some subgraph of G contains a subdivision of K_4, then so, too, does G. Without loss of generality, therefore, we may assume that G is critical, and hence that G is a block with $\delta \geq 3$. If $\nu = 4$, then G is K_4 and the theorem holds trivially. We proceed by induction on ν. Assume the theorem true for all 4-chromatic graphs with fewer than n vertices, and let $\nu(G) = n > 4$.

Suppose, first, that G has a 2-vertex cut $\{u, v\}$. By theorem 8.3, G has two $\{u, v\}$-components G_1 and G_2, where $G_1 + uv$ is 4-critical. Since $\nu(G_1 + uv) < \nu(G)$, we can apply the induction hypothesis and deduce that $G_1 + uv$

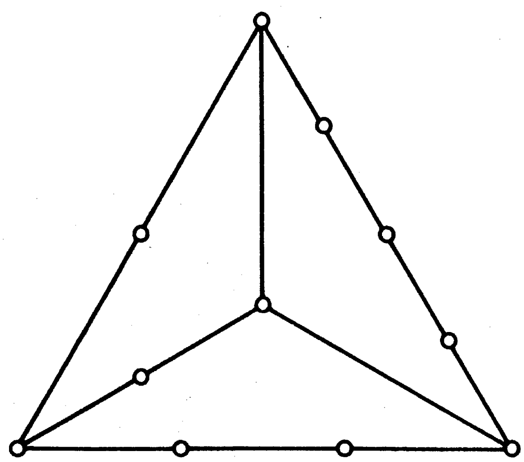


Figure 8.5. A subdivision of K_4

contains a subdivision of K_4. It follows that, if P is a (u, v)-path in G_2, then $G_1 \cup P$ contains a subdivision of K_4. Hence so, too, does G, since $G_1 \cup P \subseteq G$.

Now suppose that G is 3-connected. Since $\delta \geq 3$, G has a cycle C of length at least four. Let u and v be nonconsecutive vertices on C. Since $G-\{u, v\}$ is connected, there is a path P in $G-\{u, v\}$ connecting the two components of $C-\{u, v\}$; we may assume that the origin x and the terminus y are the only vertices of P on C. Similarly, there is a path Q in $G-\{x, y\}$ (see figure 8.6).

If P and Q have no vertex in common, then $C \cup P \cup Q$ is a subdivision of K_4 (figure 8.6*a*). Otherwise, let w be the first vertex of P on Q, and let P' denote the (x, w)-section of P. Then $C \cup P' \cup Q$ is a subdivision of K_4 (figure 8.6*b*). Hence, in both cases, G contains a subdivision of K_4 □

Hajós' conjecture has not yet been settled in general, and its resolution is known to be a very difficult problem. There is a related conjecture due to Hadwiger (1943): if G is k-chromatic, then G is 'contractible' to a graph which contains K_k. Wagner (1964) has shown that the case $k=5$ of Hadwiger's conjecture is equivalent to the famous four-colour conjecture, to be discussed in chapter 9.

Exercises

8.3.1* Show that if G is simple and has at most one vertex of degree less than three, then G contains a subdivision of K_4.

8.3.2 (*a*)* Show that if G is simple with $\nu \geq 4$ and $\varepsilon \geq 2\nu-2$, then G contains a subdivision of K_4.

(*b*) For $\nu \geq 4$, find a simple graph G with $\varepsilon = 2\nu-3$ that contains no subdivision of K_4.

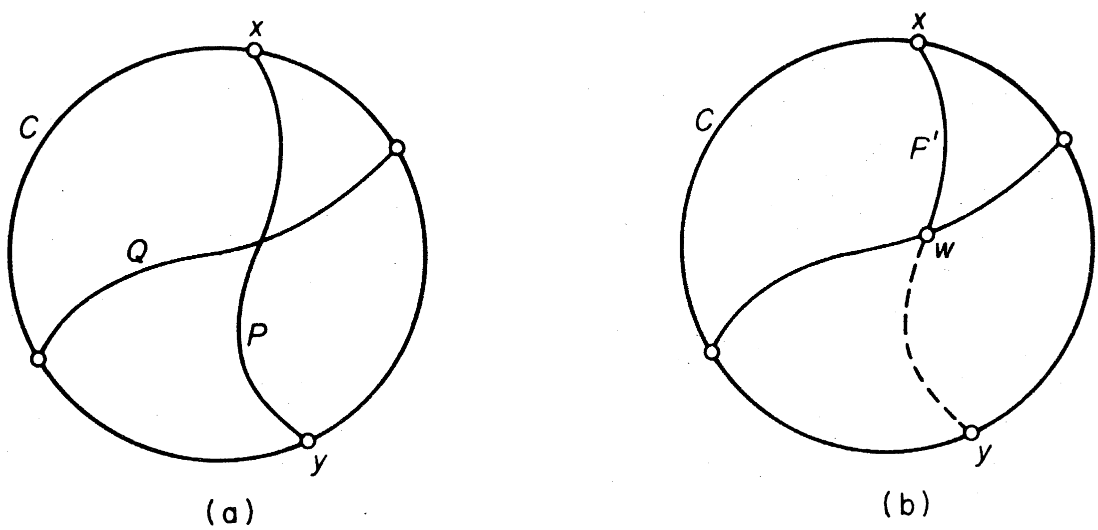


Figure 8.6

8.4 CHROMATIC POLYNOMIALS

In the study of colourings, some insight can be gained by considering not only the existence of colourings but the number of such colourings; this approach was developed by Birkhoff (1912) as a possible means of attacking the four-colour conjecture.

We shall denote the number of distinct k-colourings of G by $\pi_k(G)$; thus $\pi_k(G)>0$ if and only if G is k-colourable. Two colourings are to be regarded as distinct if some vertex is assigned different colours in the two colourings; in other words, if $(V_1, V_2, \ldots, V_k)$ and $(V_1', V_2', \ldots, V_k')$ are two colourings, then $(V_1, V_2, \ldots, V_k)=(V_1', V_2', \ldots, V_k')$ if and only if $V_i=V_i'$ for $1 \le i \le k$. For example, a triangle has the six distinct 3-colourings shown in figure 8.7. Note that even though there is exactly one vertex of each colour in each colouring, we still regard these six colourings as distinct.

If G is empty, then each vertex can be independently assigned any one of the k available colours. Therefore $\pi_k(G)=k^\nu$. On the other hand, if G is complete, then there are k choices of colour for the first vertex, $k-1$ choices for the second, $k-2$ for the third, and so on. Thus, in this case, $\pi_k(G)=k(k-1)\ldots(k-\nu+1)$. In general, there is a simple recursion formula for $\pi_k(G)$. It bears a close resemblance to the recursion formula for $\tau(G)$ (the number of spanning trees of G), given in theorem 2.8.

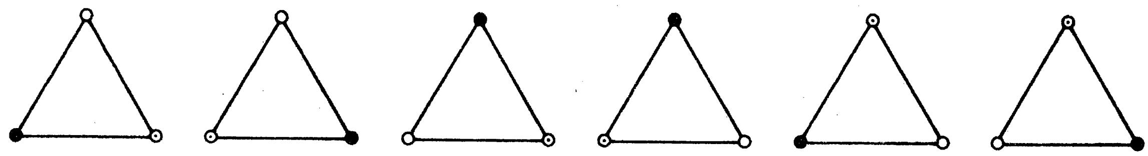


Figure 8.7

Theorem 8.6 If G is simple, then $\pi_k(G)=\pi_k(G-e)-\pi_k(G\cdot e)$ for any edge e of G.

Proof Let u and v be the ends of e. To each k-colouring of $G-e$ that assigns the same colour to u and v, there corresponds a k-colouring of $G\cdot e$ in which the vertex of $G\cdot e$ formed by identifying u and v is assigned the common colour of u and v. This correspondence is clearly a bijection (see figure 8.8). Therefore $\pi_k(G\cdot e)$ is precisely the number of k-colourings of $G-e$ in which u and v are assigned the same colour.

Also, since each k-colouring of $G-e$ that assigns different colours to u and v is a k-colouring of G, and conversely, $\pi_k(G)$ is the number of k-colourings of $G-e$ in which u and v are assigned different colours. It follows that $\pi_k(G-e)=\pi_k(G)+\pi_k(G\cdot e)$ □

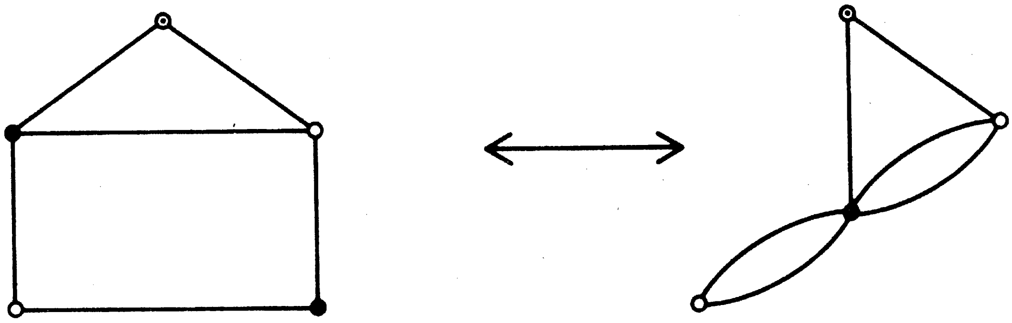


Figure 8.8

Corollary 8.6 For any graph G, $\pi_k(G)$ is a polynomial in k of degree ν, with integer coefficients, leading term k^ν and constant term zero. Furthermore, the coefficients of $\pi_k(G)$ alternate in sign.

Proof By induction on ε. We may assume, without loss of generality, that G is simple. If $\varepsilon = 0$ then, as has already been noted, $\pi_k(G) = k^\nu$, which trivially satisfies the conditions of the corollary. Suppose, now, that the corollary holds for all graphs with fewer than m edges, and let G be a graph with m edges, where $m \geq 1$. Let e be any edge of G. Then both $G-e$ and $G \cdot e$ have $m-1$ edges, and it follows from the induction hypothesis that there are non-negative integers $a_1, a_2, \ldots, a_{\nu-1}$ and $b_1, b_2, \ldots, b_{\nu-2}$ such that

$$\pi_k(G-e) = \sum_{i=1}^{\nu-1} (-1)^{\nu-i} a_i k^i + k^\nu$$

and

$$\pi_k(G \cdot e) = \sum_{i=1}^{\nu-2} (-1)^{\nu-i-1} b_i k^i + k^{\nu-1}$$

By theorem 8.6

$$\pi_k(G) = \pi_k(G-e) - \pi_k(G \cdot e)$$
$$= \sum_{i=1}^{\nu-2} (-1)^{\nu-i}(a_i + b_i)k^i - (a_{\nu-1}+1)k^{\nu-1} + k^\nu$$

Thus G, too, satisfies the conditions of the corollary. The result follows by the principle of induction □

By virtue of corollary 8.6, we can now refer to the function $\pi_k(G)$ as the *chromatic polynomial* of G. Theorem 8.6 provides a means of calculating the chromatic polynomial of a graph recursively. It can be used in either of two ways:

(i) by repeatedly applying the recursion $\pi_k(G) = \pi_k(G-e) - \pi_k(G \cdot e)$, and thereby expressing $\pi_k(G)$ as a linear combination of chromatic polynomials of empty graphs, or
(ii) by repeatedly applying the recursion $\pi_k(G-e) = \pi_k(G) + \pi_k(G \cdot e)$, and

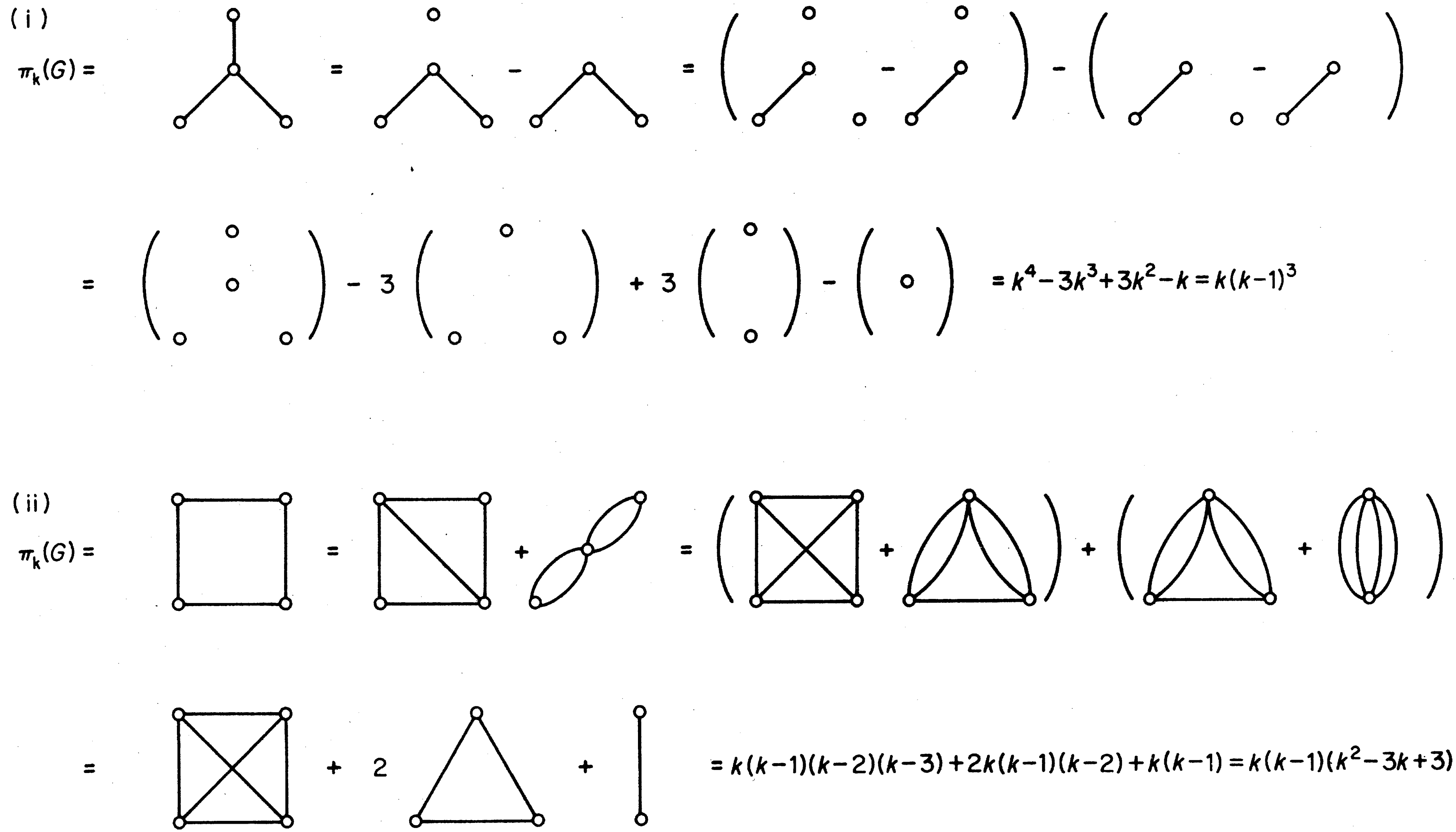


Figure 8.9. Recursive calculation of $\pi_k(G)$

thereby expressing $\pi_k(G)$ as a linear combination of chromatic polynomials of complete graphs.

Method (i) is more suited to graphs with few edges, whereas (ii) can be applied more efficiently to graphs with many edges. These two methods are illustrated in figure 8.9 (where the chromatic polynomial of a graph is represented symbolically by the graph itself).

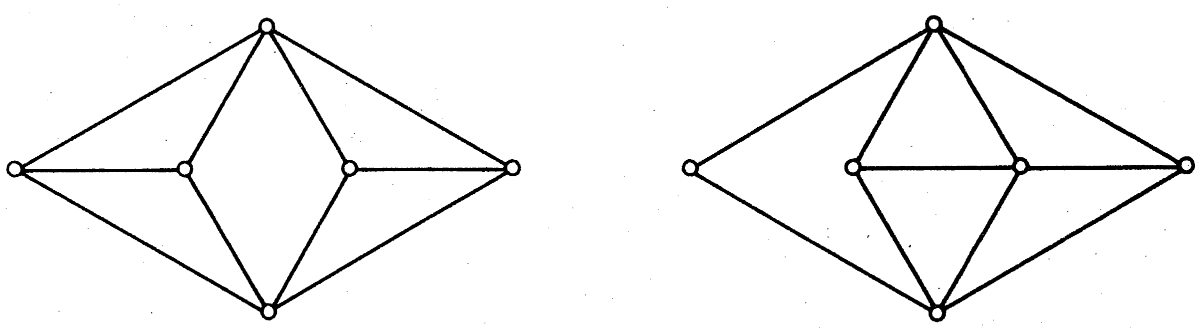
The calculation of chromatic polynomials can sometimes be facilitated by the use of a number of formulae relating the chromatic polynomial of G to the chromatic polynomials of various subgraphs of G (see exercises 8.4.5*a*, 8.4.6 and 8.4.7). However, no good algorithm is known for finding the chromatic polynomial of a graph. (Such an algorithm would clearly provide an efficient way to determine the chromatic number.)

Although many properties of chromatic polynomials are known, no one has yet discovered which polynomials are chromatic. It has been conjectured by Read (1968) that the sequence of coefficients of any chromatic polynomial must first rise in absolute value and then fall—in other words, that no coefficient may be flanked by two coefficients having greater absolute value. However, even if true, this condition, together with the conditions of corollary 8.6, would not be enough. The polynomial $k^4-3k^3+3k^2$, for example, satisfies all these conditions, but still is not the chromatic polynomial of any graph (exercise 8.4.2*b*).

Chromatic polynomials have been used with some success in the study of planar graphs, where their roots exhibit an unexpected regularity (see Tutte, 1970). Further results on chromatic polynomials can be found in the lucid survey article by Read (1968).

Exercises

8.4.1 Calculate the chromatic polynomials of the following two graphs:



8.4.2 (*a*) Show, by means of theorem 8.6, that if G is simple, then the coefficient of $k^{\nu-1}$ in $\pi_k(G)$ is $-\varepsilon$.

(*b*) Deduce that no graph has chromatic polynomial $k^4-3k^3+3k^2$.

8.4.3 (*a*) Show that if G is a tree, then $\pi_k(G)=k(k-1)^{\nu-1}$.

(*b*) Deduce that if G is connected, then $\pi_k(G)\leq k(k-1)^{\nu-1}$, and show that equality holds only when G is a tree.

8.4.4 Show that if G is a cycle of length n, then $\pi_k(G) = (k-1)^n + (-1)^n(k-1)$.

8.4.5 (*a*) Show that $\pi_k(G \vee K_1) = k\pi_{k-1}(G)$.

(*b*) Using (*a*) and exercise 8.4.4, show that if G is a wheel with n spokes, then $\pi_k(G) = k(k-2)^n + (-1)^n k(k-2)$.

8.4.6 Show that if $G_1, G_2, \ldots, G_\omega$ are the components of G, then $\pi_k(G) = \pi_k(G_1)\pi_k(G_2) \ldots \pi_k(G_\omega)$.

8.4.7 Show that if $G \cap H$ is complete, then $\pi_k(G \cup H)\pi_k(G \cap H) = \pi_k(G)\pi_k(H)$.

8.4.8* Show that no real root of $\pi_k(G)$ is greater than ν. (L. Lovász)

8.5 GIRTH AND CHROMATIC NUMBER

In any colouring of a graph, the vertices in a clique must all be assigned different colours. Thus a graph with a large clique necessarily has a high chromatic number. What is perhaps surprising is that there exist triangle-free graphs with arbitrarily high chromatic number. A recursive construction for such graphs was first described by Blanches Descartes (1954). (Her method, in fact, yields graphs that possess no cycles of length less than six.) We describe here an easier construction due to Mycielski (1955).

Theorem 8.7 For any positive integer k, there exists a k-chromatic graph containing no triangle.

Proof For $k = 1$ and $k = 2$, the graphs K_1 and K_2 have the required property. We proceed by induction on k. Suppose that we have already constructed a triangle-free graph G_k with chromatic number $k \geq 2$. Let the vertices of G_k be $v_1, v_2, \ldots, v_n$. Form a new graph G_{k+1} from G_k as follows: add $n+1$ new vertices $u_1, u_2, \ldots, u_n, v$, and then, for $1 \leq i \leq n$, join u_i to the neighbours of v_i and to v. For example, if G_2 is K_2 then G_3 is the 5-cycle and G_4 the Grötzsch graph (see figure 8.10).

The graph G_{k+1} clearly has no triangles. For, since $\{u_1, u_2, \ldots, u_n\}$ is an independent set in G_{k+1}, no triangles can contain more than one u_i; and if $u_iv_jv_ku_i$ were a triangle in G_{k+1}, then $v_iv_jv_kv_i$ would be a triangle in G_k, contrary to assumption.

We now show that G_{k+1} is $(k+1)$-chromatic. Note, first, that G_{k+1} is certainly $(k+1)$-colourable, since any k-colouring of G_k can be extended to a $(k+1)$-colouring of G_{k+1} by colouring u_i the same as v_i, $1 \leq i \leq n$, and then assigning a new colour to v. Therefore it remains to show that G_{k+1} is not k-colourable. If possible, consider a k-colouring of G_{k+1} in which, without loss of generality, v is assigned colour k. Clearly, no u_i can also have colour k. Now recolour each vertex v_i of colour k with the colour assigned to u_i.

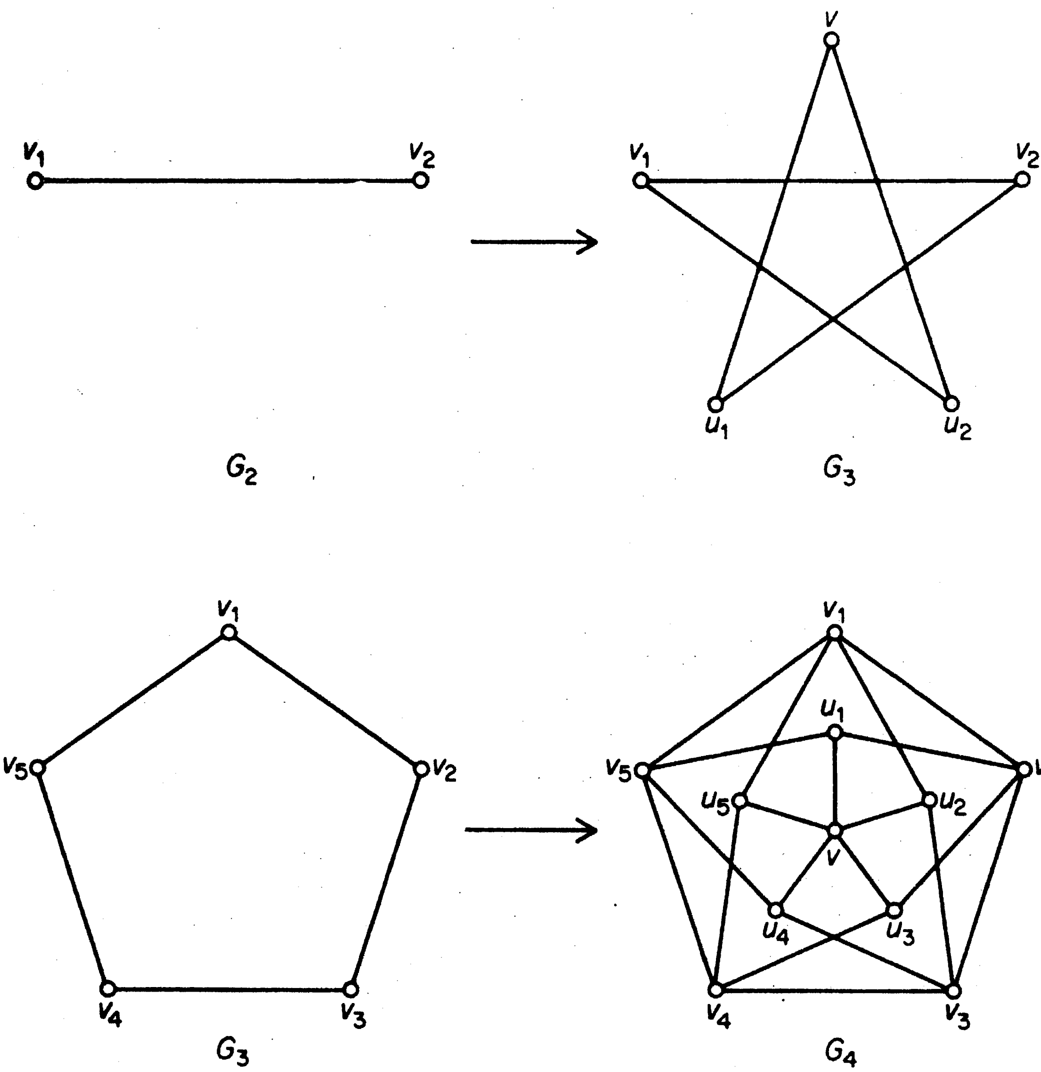


Figure 8.10. Mycielski's construction

This results in a $(k-1)$-colouring of the k-chromatic graph G_k. Therefore G_{k+1} is indeed $(k+1)$-chromatic. The theorem follows from the principle of induction □

By starting with the 2-chromatic graph K_2, the above construction yields, for all $k \geq 2$, a triangle-free k-chromatic graph on $3.2^{k-2}-1$ vertices.

We have already noted that there are graphs with girth six and arbitrary chromatic number. Using the probabilistic method, Erdös (1961) has, in fact, shown that, given any two integers $k \geq 2$ and $l \geq 2$, there is a graph with girth k and chromatic number l. Unfortunately, this application of the probabilistic method is not quite as straightforward as the one given in section 7.2, and we therefore choose to omit it. A constructive proof of Erdös' result has been given by Lovász (1968).

Exercises

8.5.1 Let $G_3, G_4, \ldots$ be the graphs obtained from $G_2 = K_2$, using Mycielski's construction. Show that each G_k is k-critical.

8.5.2 (*a*)* Let G be a k-chromatic graph of girth at least six ($k \geq 2$). Form a new graph H as follows: Take $\binom{k\nu}{\nu}$ disjoint copies of G and a set S of $k\nu$ new vertices, and set up a one–one correspondence between the copies of G and the ν-element subsets of S. For each copy of G, join its vertices to the members of the corresponding ν-element subset of S by a matching. Show that H has chromatic number at least $k+1$ and girth at least six.

(*b*) Deduce that, for any $k \geq 2$, there exists a k-chromatic graph of girth six. (B. Descartes)

APPLICATIONS

8.6 A STORAGE PROBLEM

A company manufactures n chemicals $C_1, C_2, \ldots, C_n$. Certain pairs of these chemicals are incompatible and would cause explosions if brought into contact with each other. As a precautionary measure the company wishes to partition its warehouse into compartments, and store incompatible chemicals in different compartments. What is the least number of compartments into which the warehouse should be partitioned?

We obtain a graph G on the vertex set $\{v_1, v_2, \ldots, v_n\}$ by joining two vertices v_i and v_j if and only if the chemicals C_i and C_j are incompatible. It is easy to see that the least number of compartments into which the warehouse should be partitioned is equal to the chromatic number of G.

The solution of many problems of practical interest (of which the storage problem is one instance) involves finding the chromatic number of a graph. Unfortunately, no good algorithm is known for determining the chromatic number. Here we describe a systematic procedure which is basically 'enumerative' in nature. It is not very efficient for large graphs.

Since the chromatic number of a graph is the least number of independent sets into which its vertex set can be partitioned, we begin by describing a method for listing all the independent sets in a graph. Because every independent set is a subset of a maximal independent set, it suffices to determine all the maximal independent sets. In fact, our procedure first determines complements of maximal independent sets, that is, minimal coverings.

Observe that a subset K of V is a minimal covering of G if and only if, for each vertex v, either v belongs to K or all the neighbours of v belong to K (but not both). This provides us with a procedure for finding minimal coverings:

FOR EACH VERTEX v, CHOOSE EITHER v, OR ALL THE NEIGHBOURS OF v

(8.2)

To implement this procedure effectively, we make use of an algebraic device. First, we denote the instruction 'choose vertex v' simply by the symbol v. Then, given two instructions X and Y, the instructions 'either X or Y' and 'both X and Y' are denoted by $X+Y$ (the *logical sum*) and XY (the *logical product*), respectively. For example, the instruction 'choose either u and v or v and w' is written $uv+vw$. Formally, the logical sum and logical product behave like $\cup$ and $\cap$ for sets, and the algebraic laws that hold with respect to $\cup$ and $\cap$ also hold with respect to these two operations (see exercise 8.6.1). By using these laws, we can often simplify logical expressions; thus

$$\begin{aligned}(uv+vw)(u+vx) &= uvu+uvvx+vwu+vwvx\\ &= uv+uvx+vwu+vwx\\ &= uv+vwx\end{aligned}$$

Consider, now, the graph G of figure 8.11. Our prescription (8.2) for finding the minimal coverings in G is

$$(a+bd)(b+aceg)(c+bdef)(d+aceg)(e+bcdf)(f+ceg)(g+bdf) \quad (8.3)$$

It can be checked (exercise 8.6.2) that, on simplification, (8.3) reduces to

$$aceg+bcdeg+bdef+bcdf$$

In other words, 'choose a, c, e and g or b, c, d, e and g or b, d, e and f or b, c, d and f'. Thus $\{a, c, e, g\}$, $\{b, c, d, e, g\}$, $\{b, d, e, f\}$ and $\{b, c, d, f\}$ are the minimal coverings of G. On complementation, we obtain the list of all maximal independent sets of G: $\{b, d, f\}$, $\{a, f\}$, $\{a, c, g\}$ and $\{a, e, g\}$.

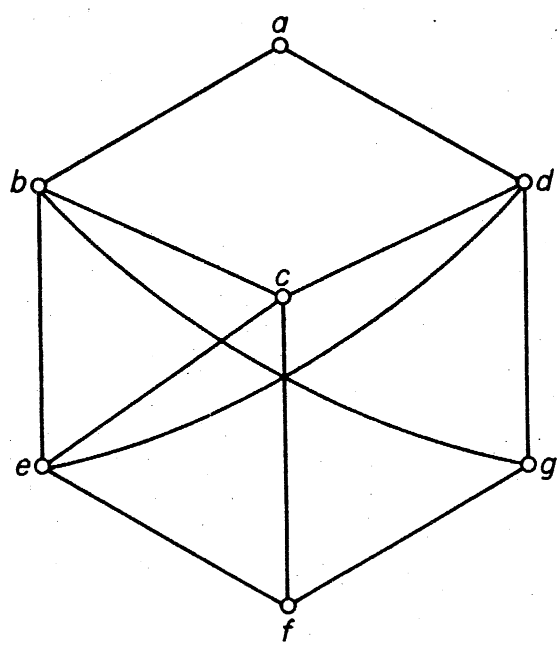


Figure 8.11

Now let us return to the problem of determining the chromatic number of a graph. A k-colouring $(V_1, V_2, \ldots, V_k)$ of G is said to be *canonical* if V_1 is a maximal independent set of G, V_2 is a maximal independent set of $G - V_1$, V_3 is a maximal independent set of $G-(V_1 \cup V_2)$, and so on. It is easy to see (exercise 8.6.3) that if G is k-colourable, then there exists a canonical k-colouring of G. By repeatedly using the above method for finding maximal independent sets, one can determine all the canonical colourings of G. The least number of colours used in such a colouring is then the chromatic number of G. For the graph G of figure 8.11, $\chi = 3$; a corresponding canonical colouring is $(\{b, d, f\}, \{a, e, g\}, \{c\})$.

Christofides (1971) gives some improvements on this procedure.

Exercises

8.6.1 Verify the associative, commutative, distributive and absorption laws for the logical sum and logical product.

8.6.2 Reduce (8.3) to $aceg + bcdeg + bdef + bcdf$.

8.6.3 Show that if G is k-vertex-colourable, then G has a canonical k-vertex colouring.

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